Stability Analysis of ZND Detonation for Majda's Model with More General Ignition Function

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1 Introduction

The Euler equation is usually used to study the detonation performance. Euler equations contain mass conservation equation, energy conservation equation, momentum conservation equation and reaction rate equation. Excessive variables in the Euler equation make calculations and theoretical analysis more difficult. Majda model[1-3] simplifies the Euler equations. Majda model can capture many of the phenomena of the much more complicated reactive Euler and Navier-Stokes equations governing physical detonation and eliminate enough of technical complexities so as to be mathematically easy.

The Majda model contain a Burgers equation and a chemical kinetics equation[4-5]:

$$\begin{cases}(u+qz)_t + f(u)_x = \varepsilon u_{xx} \\ z_t + k\varphi(u)z = 0\end{cases} \tag{1}$$

Here, subscript denote differentiate with respect to x and t; u is a lumped scalar variable representing various aspects of density, pressure, and temperature; z is the mass fraction of reactant in a simple one-step reaction scheme ,if no reaction occurred in front of shock wave, so z=1. When the reaction is over, z=0, $f(u)$is a nonlinear convex function; q, k, ε are positive constants measuring reaction rate, heat release, and viscosity, respectively. $\varphi(u)$ is the ignition function which turns on the reaction.

If set:

$$\varepsilon = 0 \quad f(u) = \frac{u^2}{2}, \quad \varphi(u,z) = \begin{cases}0 & u \le u_{ig} \\ e^{\left[\theta\left(\sqrt{q}u + q(1-z)\right)\right]} & u > u_{ig}\end{cases} \tag{2}$$

Where, the subscript "*ig*" means "ignition" .

The Majda model become as follow:

$$\begin{cases} \left(u+qz\right)_t + \left(\dfrac{u^2}{2}\right)_x = 0 \\ z_t + k\varphi\left(u,z\right)z = 0 \\ \varphi\left(u,z\right) = \begin{cases} 0 & u \le u_{ig} \\ e^{\left[\theta\left(\sqrt{q}u+q(1-z)\right)\right]} & u > u_{ig} \end{cases} \end{cases} \tag{3}$$

Here, u can represent density, pressure, and temperature; z_t simulates the reaction rate in the Euler system; θ simulate reaction activation energy.

Above model is a Majda simplified model for studying the stability of ZND detonation in this paper. This model is simpler than Euler equation. In the model , the chemical reaction equation is an Arrhenius reaction and ignition equation is more general and realistic than previous work .

2 Detonation stability theory for Majda model

The leading shock wave is considered to be a shock wave front that superimposes the perturbation on a steady state. If the perturbation is attenuated for every frequency and wavelength, the detonation structure is stable. If there is any perturbation that increases with time, the detonation wave structure is considered unstable.

According to the Stability criterion[6-9], we need to know the real part of the eigenvalues when the stability function Eq.(4) is equal to zero. If there is at least one eigenvalue's real part is greater than zero, the detonation is unstable. If the real part of eigenvalues are all smaller than zero, the detonation is stable.

$$D_{ZND}\left(\lambda\right) = \det\left(Z^{-}\left(\lambda,0\right), \lambda\left[\bar{W}^0\right] - \left[A\left(\bar{W}^0\right)'\right]\right) = 0 \tag{4}$$

$D_{ZND}(\lambda)$is the stability function Evans-Lopatinski determinant , λ is eigenvalue. $W = \left(u \quad z\right)^T$, $\bar{W} = \left(\bar{u} \quad \bar{z}\right)^T$ is the steady solution. $\left(u , z\right)\left(x , t\right) = \left(\bar{u} , \bar{z}\right)\left(x - st\right)$, s is speed of shock wave.

When ignition function $\phi\left(u\right)$ is not simple enough (such as Heaviside function), it is very difficult to get the analytical expression of Z for Eq.(3), it is very difficult to solve Eq.(4)directly. Therefore, we use the winding number for stability analysis in this paper.

Winding number[10] is a method for determining whether there is a zero in a certain area in the complex plane. According to the stability criterion, as long as the real part of eigenvalue is greater than zero, the detonation is unstable. So, we only need to find the eigenvalues λ that satisfy $D_{ZND}\left(\lambda\right) = 0$ in the complex plane where the real part is greater than zero($\mathrm{Re}\left(\lambda\right) > 0$). If there is an eigenvalue with $\mathrm{Re}\,\lambda > 0$, the detonation is unstable.

The number of roots indicates the number of eigenvalues which satisfies $D_{ZND}\left(\lambda\right) = 0$ in unstable region. We give the semicircle figure 1:

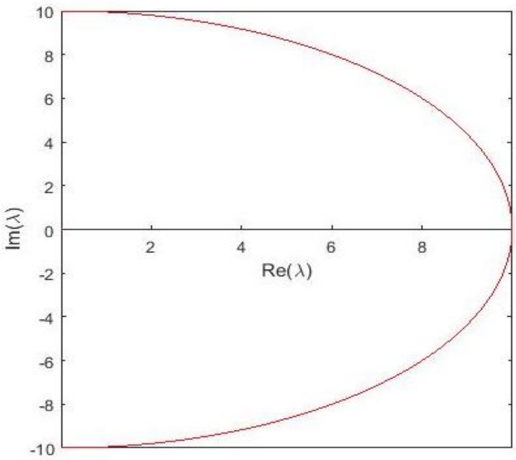

Figure 1: The semicircle of radius R

Substituting every λ on the boundary of semicircle into Eq. (4), we will obtain a figure of $D_{ZND}(\lambda)$. We can determine the winding number from the figure of $D_{ZND}(\lambda)$. The winding number is the number roots of $D_{ZND}(\lambda)=0$ in the semicircular area of radius *R*. If the winding number is not zero, it means there are eigenvalues with $\mathrm{Re}(\lambda)>0$ in semicircular area of radius *R*. So detonation is unstable. If the winding number is zero in semicircular area of radius *R*, it means there is no eigenvalue with $\mathrm{Re}(\lambda)>0$. So it is stable in semicircular area of radius *R*.

3 Example of stability analysis for Majda's Model

The stability analysis of Eq.(3) by the winding number for different parameters is conducted.

The first group parameters: $q=0.5, k=1$, $\theta=1$, 2, 4, 8, the results are following:

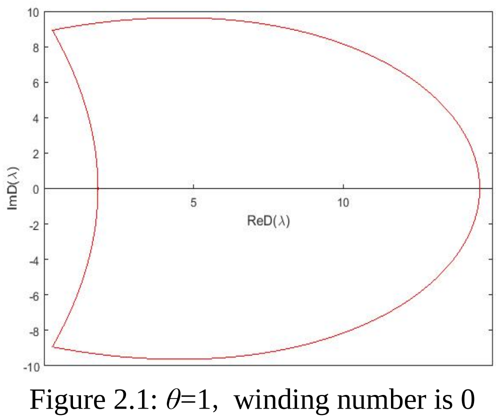

Figure 2.1: θ=1, winding number is 0

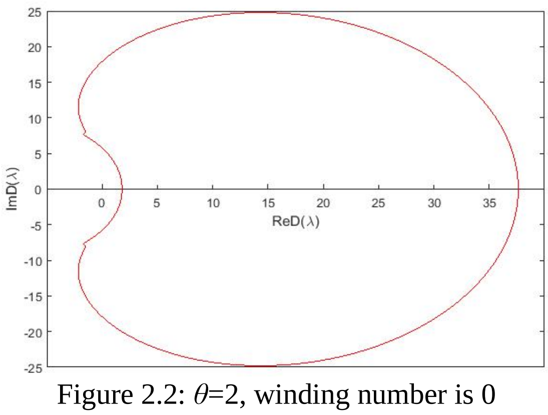

Figure 2.2: θ=2, winding number is 0

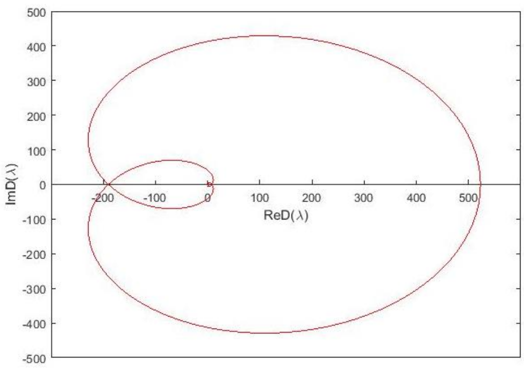

Figure 2.3: θ=4, winding number is 2

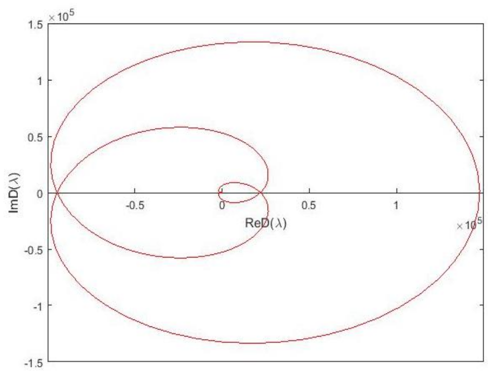

Figure 2.4: θ=8, winding number is 4

In the case of parameters $q = 0.5, k = 1$, when the simulated activation energy $\theta = 1, 2$, the winding number is zero in semicircular area of radius *R*=10. Therefore, the detonation is stable in the selected area *R*=10. When the simulated activation energy $\theta = 4$, the winding number is 2 , when $\theta = 8$, the winding number is 4; the detonation are unstable.

The second group parameters: $q = 0.3, k = 1$, $\theta = 1$, 2, 10, 12, the result is as follow:

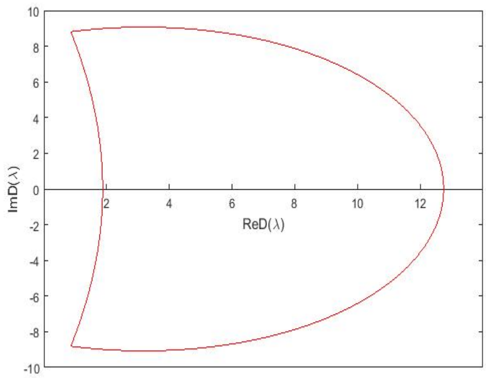

Figure 3.1: θ=1, winding number is 0

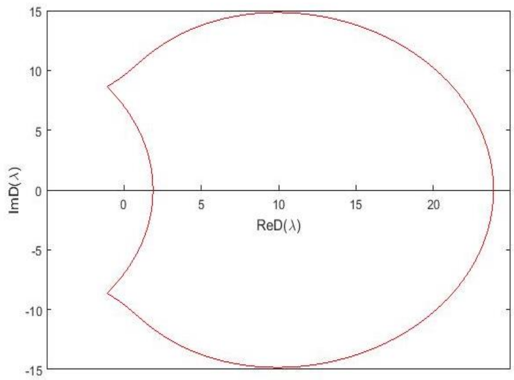

Figure 3.2: θ=2, winding number is 0

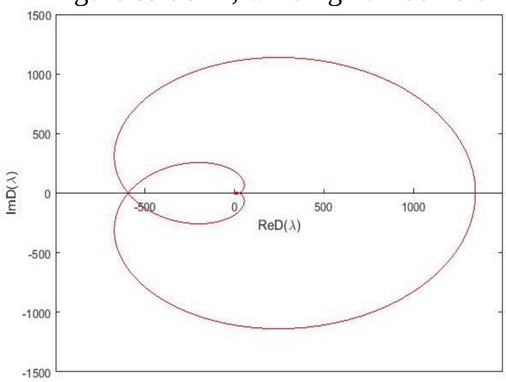

Figure 3.3: θ=10, winding number is 2

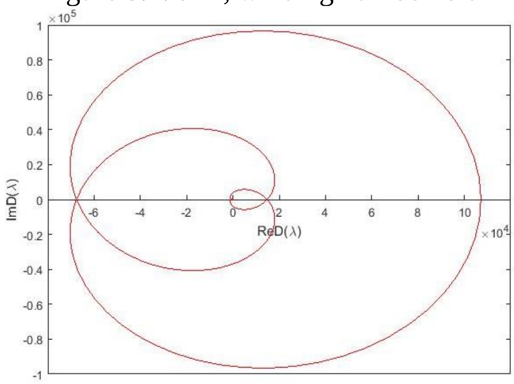

Figure 3.4: θ=12, winding number is 4

When the simulated activation energy $\theta = 1, 2$, the winding number is 0 in semicircular area of radius *R*=10. Therefore, the detonation is stable in the selected area.

When the simulated activation energy $\theta = 10$, the result of winding number is 2; when $\theta = 12$, the winding number is 4. Therefore, the detonation is unstable .

From Fig. 2 and Fig.3, we can see: for smaller θ ,the detonation is stable, for larger θ ,the detonation tends to be unstable. The physical reason is that: low activation energy θ means low temperature sensitivity for reaction rate, which makes the detonation more stable.

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