Stability analysis of the Noh problem for reactive shocks

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1 Introduction

The studies of shock compression of condensed media and shock-front stability started simultaneously in the 1940s within nuclear weapons projects. Notwithstanding the impressive progress made in both fields since then, the fundamental shock-front instability theoretically discovered by D'yakov (1954) [1] and Kontorovich (1957) [2] (DK) in the USSR still challenges our understanding of shock compression in condensed media. DK predicted non-decaying oscillations of an isolated planar shock front, accompanied by spontaneous acoustic emission. This subtle effect is only possible under strict constraints on the equation of state and shock strength resulting in a specific shape of the Hugoniot curve. It took 20 years since the DK discovery to find realistic shock-compression conditions for its manifestation [3] and many more years until its first numerical demonstration in van der Waals fluid [4]. The discussion about the nature of this elusive phenomenon continues in the literature to this day.

Steady shocks can exist if there is a supporting mechanism that maintains downstream pressure constant. The same applies for overdriven detonations or endothermic shocks. If the shock front is over-supported, it is followed by a compression wave that gradually increases its strength, as occurs in Guderley-type converging shocks (Guderley 1942) [5]. By contrast, if the shock is under-supported because it is followed by an expansion wave, its intensity gradually diminishes, as found in Sedov-type blast waves [6, 7]. The flow associated with these two cases can be unstable even when the shock front *per se* is surely stable (blast wave [8, 9] and converging shock [10, 11] in an ideal gas are examples). Then, in order to study the stability of the shock wave in a manner that results can be attributed to the shock itself, the shock front must be steady and the downstream flow must be constant.

For a planar shock, the steady state can be simply given, but not uniquely, by the consideration of a rigid piston that moves at constant speed and keeps downstream pressure constant. Self-sustained detonations are another example. A spherically or cylindrically expanding shock front, just like a planar one, can be steady if a constant-velocity inflow of incident mass supports it [7,12,13]. In such case, the center or axis of symmetry plays the role of the rigid piston and the shock strength, characterized by velocity, pressure and density jumps across the shock, is time independent, thereby yielding uniform properties of the shocked gas put at rest. Such an expanding accretion-shock flow is described by a self-similar solution of ideal compressible fluid dynamics equations, which exists for an arbitrary EoS of the fluid and the parameters of the incident flow [12, 13]. The inclusion of endothermic and exothermic effects across the shock, which is of interest in many high-energy-density contexts, does not modify this property although it involves additional restrictions on the admissibility of solutions of the Riemann problem.

What is known on the stability of steady isolated shock can be summarized as follows. The stability of the shock depends the DK parameter

$$h = \frac{p_2 - p_1}{V_1 - V_2} \left(\frac{\mathrm{d}V_2}{\mathrm{d}p_2} \right)_H = -u_2^2 \left(\frac{\partial \rho_2}{\partial p_2} \right)_H \tag{1}$$

that measures the slope of the Hugoniot curve relative to the Rayleigh-Michelson line on the (V, p) plane. Here p, ρ, $V = 1/\rho$, and u denote the pressure, density, specific volume, and fluid velocity with respect to the shock front, respectively, subscripts 1 and 2 refer to pre- and post-shock states, and the derivatives are calculated along the Hugoniot curve with the pre-shock pressure and density fixed. For an isolated steady planar shock front, the classic stability theory predicts an oscillatory decay of perturbations as $t^{-3/2}$ ($t^{-1/2}$ in the strong-shock limit), with a constant oscillation frequency, for any wavenumber and any EoS [14], provided that the parameter h is in the stable range, $-1 < h < h_c$, where

$$h_c = \frac{1 - \mathcal{M}_2^2 (1 + \mathcal{R})}{1 - \mathcal{M}_2^2 (1 - \mathcal{R})}, \tag{2}$$

and $\mathcal{R} = \rho_2/\rho_1$ is the shock density compression ratio. For $h_c < h < 1 + 2\mathcal{M}_2$, shock perturbations with certain wavevectors oscillate at constant amplitude, causing spontaneous acoustic emission (SAE) from the shock front [2, 15–17]. Absolutely unstable ranges are $h < -1$ and $h > 1 + 2\mathcal{M}_2$, for which the exponential growth of shock-front perturbations is associated with conditions that render multi-valued [18, 19] or multi-wave [20, 21] solutions of the planar Riemann/piston problem.

When considering a supporting wall, like a moving piston, the radiated acoustic waves reverberate, thereby involving additional frequencies to the oscillating shock dynamics. Reflected waves arrive at the shock with a lower frequency by the Doppler shift effect. In the stable range, $-1 < h < h_c$, the acoustic disturbances decay exponentially with the distance from the shock, which implies that the acoustic coupling is very weak, even for short periods of time. This is the reason why stability threshold and the decay rate of the shock oscillations amplitude in the stable range takes the same form in both isolated and piston-driven configurations [22–24]. Likewise, the absolutely unstable ranges $h < -1$ and $h > 1 + 2\mathcal{M}_2$, which are exclusively associated with the Rankine-Hugoniot curve, are the same regardless the boundary condition. By way of contrast, including a supporting mechanism can make a difference in the marginally stable/SAE range, $h_c < h < 1 + 2\mathcal{M}_2$, since the acoustic radiation does not decay with the distance from the shock and the acoustic coupling is always present. As noted in Refs. [19, 24], the amplitude of a reverberating acoustic wave grows as a power of time for $h > 1$. The stability analysis is not closed for $h_c < h < 1$. While Ref. [25] did not find any qualitative distinctness in the shock front perturbation behavior when a piston is involved for $h_c < h < 1 - 2\mathcal{M}_2^2$, Ref. [26] found instability, a linear growth of shock perturbations in the whole range $h > h_c$. A more recent study [13] based on the Noh problem suggests that $h > h_c$ may lead to instability with a perturbation growth rate that depends on the shock properties. In this work we present a similar analysis as in Ref. [13] but extended to endothermic or exothermic shocks.

2 Formulation of the reactive Noh problem

According to equations (1) and (2), the DK stability boundary for a planar shock depends on three parameters: the shock compression ratio $\mathcal{R}$, the post-shock Mach number $\mathcal{M}_2$ and the slope of the RH curve through the parameter h. They can be constructed by integrating the conservation equations across the shock. In particular, if we admit the possibility of having exothermic or endothermic effects throughout the shock wave, the conservation equations read as

$$\rho_1 u_1 = \rho_2 u_2, \tag{3a}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2, \tag{3b}$$

$$p_1/\rho_1 + E_1 + u_1^2/2 + q = p_2/\rho_2 + E_2 + u_2^2/2, \tag{3c}$$

where p is the thermal pressure, E is the internal energy and q measures the energy removed ($q < 0$) or released ($q > 0$) per unit mass by the corresponding process undergoing across the shock. For simplicity, we restrict ourselves to $q =$constant, although it is known that variable q (more likely in endothermic cases) could render different results [27].

We are considering an extension of the generalized Noh problem, where $E(p, \rho)$ is a known function of pressure and density with no particular restriction on its functional dependence. In our case, the vdW equation of state is employed to provide a more accurate description of real fluid behaviour than the ideal gas law. In this case, the speed of sound and internal energy read

$$c^2 = \frac{\gamma(p + \rho^2 a)}{\rho(1 - b\rho)} - 2a\rho \quad \text{and} \quad E = \frac{(p + \rho^2 a)(1 - b\rho)}{\rho(\gamma - 1)} - a\rho, \tag{4}$$

respectively, where γ is the adiabatic index. The parameters a and b have positive values and are specific to each gas. With respect to the ideal gas equation of state, the term involving the constant a corrects for intermolecular attraction, while b represents the volume occupied by the gas particles.

The problem is initiated as follows. At $t = 0^-$, the fluid has uniform density $\rho_1(\boldsymbol{r}, t = 0) = \rho_0$, pressure $p_1(\boldsymbol{r}, t = 0) = p_0$, and velocity $\boldsymbol{v}_1(\boldsymbol{r}, t = 0) = -v_0 \boldsymbol{e}_r$ directed to the center or axis of symmetry. At $t = 0^+$, an accretion shock emerges from the origin. The lack of scales justifies the definition of the self-similar coordinate $\xi = r/v_0 t$ to describe the converging flow. Then, the mass, radial momentum, and energy conservation equations are rewritten as a system of differential equations for ξ, namely

$$(v_1 - \xi v_0)\frac{\mathrm{d}\ln\rho_1}{\mathrm{d}\xi} + \frac{\mathrm{d}v_1}{\mathrm{d}\xi} + \frac{(\nu - 1)v_1}{\xi} = 0, \tag{5a}$$

$$(v_1 - \xi v_0)\frac{\mathrm{d}v_1}{\mathrm{d}\xi} + \frac{1}{\rho_1}\frac{\mathrm{d}p_1}{\mathrm{d}\xi} = 0, \tag{5b}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}\xi} - c_1^2\frac{\mathrm{d}\rho_1}{\mathrm{d}\xi} = 0, \tag{5c}$$

The coefficient ν represents the geometry, where $\nu = 1$ is the planar geometry that renders a trivial flat-profile behaviour, and $\nu = 2$ and $\nu = 3$ refer to the cylindrical and spherical geometries, respectively, which provide a variable flow as a result of the inwards mass accumulation. Subscript 1 refers to variable conditions in the whole domain ahead of the shock front while subscript 0 indicates the initial conditions.

Equations (5a)-(5c) are supplemented with the equation for the speed of sound $c_1 = c(p_1, \rho_1)$, to be determined with (4), and the associated boundary conditions $v_1(\xi \to \infty) - v_0 = \rho_1(\xi \to \infty) - \rho_0 = p_1(\xi \to \infty) - p_0 = 0$ that impose the properties far from the center at a given time frame $r \gg v_0 t$.

3 Linear stability analysis

For a spherical geometry, the perturbed shock-front position $\delta r_s = r_s(\theta, \varphi, t) - v_s t$ is written in terms of spherical harmonics, i.e.,

$$\frac{\delta r_s}{v_s t} = \epsilon \sum_{l,m} \zeta_{l,m} \left(\frac{t}{t_0}\right)^{\sigma_{l,m}} Y_l^m(\theta, \varphi), \tag{6}$$

where v_s and $v_s t$ correspond to the unperturbed shock velocity and radial position of the shock, respectively, and $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ stand for the polar and azimuthal angles, respectively. The term proportional to the small-amplitude parameter $\epsilon \ll 1$ includes $Y_l^m(\theta, \varphi) = P_l^m(\cos\theta)\exp(\mathrm{i}m\varphi)$, where P_l^m is the associated (generalized) Legendre function with l and $m \leq l$ referring to the polar and azimuthal integer mode numbers. For each mode may correspond a different amplitude, $\zeta_{l,m}$. The lack of scales dictates the power-law dependence $(t/t_0)^\sigma$, where t_0 is an arbitrary temporal parameter used to provide dimensional consistency and $\sigma_{l,m} = \sigma = \sigma_\mathrm{R} + \mathrm{i}\sigma_\mathrm{I}$ is the complex dimensionless eigenvalue.

The integration of the Euler equations can be done analytically following the procedure detailed in Refs. [13] which can be summarized as follows. Perturbation variables are functions of the corresponding eigenfunction, which depends on the self-similar variable ξ, times the dependence $(t/t_0)^\sigma$. The pressure eigenfunction is only of acoustic type, while the density and velocity eigenfunctions are split into acoustic and entropic for the former and acoustic and vortical for the latter. The acoustic mode corresponds to standing waves whose amplitude depends on the shock pressure. On the other hand, the entropic and vortical perturbations are steady functions whose value depends on that left by the shock front at the location $r = v_s t$. Finally, the integration of the Euler equation along with the shock boundary conditions provides the value of the three mode amplitudes and the eigenvalue σ, the latter being determined by the dispersion relationship

$$\frac{\mathcal{R}(\nu-1) - \sigma - \nu + \mathcal{R}j(j+\nu-2)}{\sigma+\nu+j-1} F_{1s}^{+} = \frac{\mathcal{R}(\nu-1)\,(1+h_1) - 2(\sigma+\nu)}{(1+h)\,(\sigma+\nu-1)} F_{1s}^{-}, \tag{7}$$

where the functions F_{1s}^{+} and F_{1s}^{-}, defined conjointly as

$$F_{1s}^{\pm} = {}_2F_1\left(\frac{j-\sigma}{2}, \frac{j\pm 1-\sigma}{2}; j+\frac{\nu}{2}; \eta^2 = \mathcal{M}_2^2\right), \tag{8}$$

refer to the Gauss hypergeometric functions evaluated at the shock front. The influence of the upstream variations is accounted in the factor h_1 that does not appear in the planar shock case. This parameter is proportional to h but also includes the derivatives with respect to density and pressure upstream. From (7) we see that h_1 is less important the higher is the mode number j. In particular, if we define h_{st} as the lowest value of h that makes $\sigma_\mathrm{R} = 0$ in (7), we get $h_{st}(\mathcal{M}_2, \mathcal{R}, h_1, \nu, j) \to h_c(\mathcal{M}_2, \mathcal{R})$ when $j \to \infty$, for any set of shock conditions (exothermicity/endothermicity) and geometrical properties (spherical/cylindrical), as should occur.

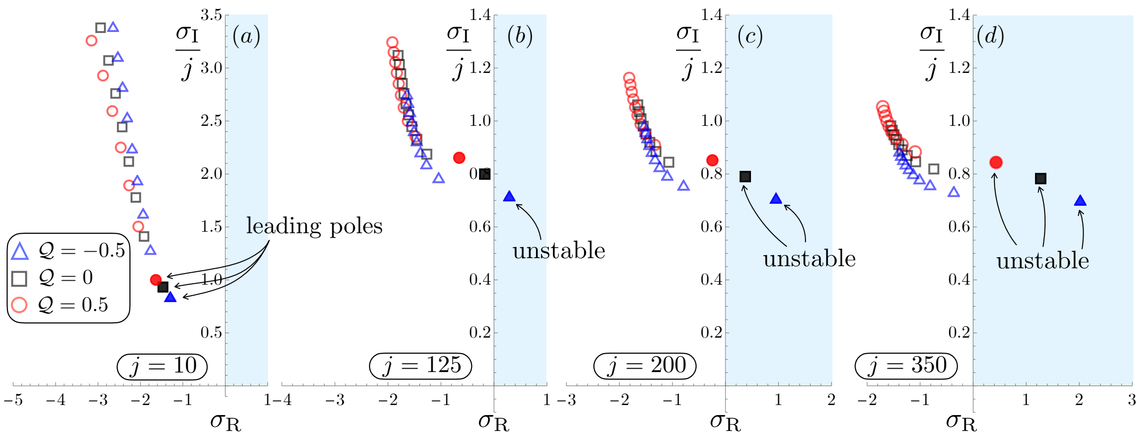

Figure 1: Eigenvalues for an expanding cylindrical shock ($\nu = 2$) with $\mathcal{R} = 2.9$ in a vdW EoS with $\gamma = 31/30$ with $\alpha_1 = 1/2$, and $\beta_1 = 1/9$ and different values energy $\mathcal{Q} = -0.5$, 0 and 0.5. Four different mode numbers are chosen: $j = 10$, $j = 125$, $j = 200$, and $j = 350$.

In order to study the effect of endothermicity and exothermicity in the unstable range $h > h_c$ for finite numbers of j, Fig.1 displays the first seven (lowest frequencies) values of σ with three different values of the dimensionless energy factor $\mathcal{Q} = q\rho_{1s}/p_{1s}$. We choose the same gas properties as those employed by Refs. [4, 13], associated with high compressibility. Mode numbers are purposely chosen to give four different results: all are stable cases; only the endothermic case is unstable; only the exothermic case is stable; and all are unstable cases. It is observed that shocks are all stable for sufficiently low mode numbers, even if condition $h > h_c$ is satisfied. This is because of the stabilizing effect of the expanding domain. Then, within the unstable range $h > h_c$, it is the mode number what determines the instability threshold. As j increases, the set of complex eigenvalues moves towards the half-plane $\sigma_{\rm R} > 0$, being the one with the lowest oscillation frequency what crosses the axis $\sigma_{\rm R} = 0$ first, thereby setting the stability threshold $h = h_{st}$. If we consider a case with $h < h_c$ instead, $\sigma_{\rm R} < 0$ for any value of j.

4 Conclusions

It has been studied the influence of endothermicity/exothermicity on the stability of expanding shocks within the Noh problem configuration. An unexpected finding in that exothermicity reduces the unstable range when compared to the adiabatic case. Further computations reveal that a sufficiently exothermic shock can make the shock stable in the whole domain of shock intensities. By way of contrast, endothermicity modifies the unstable range differently: for low energy substraction the unstable range is expanded, but for high endothermic processes it splits into two. This conclusions may be altered when considering endothermic or exothermic phenomena that are shock-strength dependent, as may be gas dissociation and ionization or thermonuclear detonations, respectively.

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