Analog modeling of detonation in a periodic medium

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1 Introduction

Wave propagation in heterogeneous media is associated with a number of complex phenomena that have received considerable attention in the past. Most of the existing research is concerned with linear waves, and that already poses substantial theoretical and numerical challenges. Understanding shock and detonation waves in heterogeneous media is also of interest [4, 5]. The heterogenous medium in prior studies usually consisted of layers or random packings of discrete components. For the sake of simplicity, in this work we consider a medium with properties varying smoothly in a spatially periodic fashion, and analyze detonation in such a medium employing an analog model. Our main question is: How are intrinsic instabilities of detonation, that exist even in the absence of heterogeneity, influenced by the periodic variations of properties of the reactive medium?

The analysis is carried out with the appropriately generalized reactive Burgers equation of [3]:

$$u_t + 0.5(u^2 - uu_s)_x = f(x, u_s), \quad x < 0, t > 0, \tag{1}$$

where $u(x, t)$ is the unknown function (analog of pressure, for example), subscripts t and x denote partial derivatives with respect to time and space, respectively, $u_s = u(0, t)$ is the value of u at the shock, $x = 0_-$. Equation 1 is written in a shock-attached frame for the wave that propagates from left to right. The reaction zone is at $x < 0$, where $f(x, u_s(t))$ acts as an analog of a reaction source term. Note that in all previous studies of this model [1–3], the upstream state, $u(x > 0, t)$, was always assumed to be uniform and given by $u = u_a = 0$. In this work, we take a more general u_a which is either a non-zero constant or a spatially periodic smooth function.

We find that the upstream state influences both the stability of the steady-state traveling wave solutions (i.e. ZND solutions) and the character of the ensuing instabilities. For example, increasing the magnitude of $u_a = const$ above zero is found to have a stabilizing effect on the solutions. In the more interesting case of a periodically varying u_a, we find a variety of modes of interaction between the intrinsic dynamics of the wave and the periodic forcing arising from the upstream-state nonuniformities. In particular, there exists a kind of a resonance whereby the amplitude of the shock pulsations increases due to the presence of upstream oscillations compared to the case of a uniform state with the same average as the corresponding non-uniform state.

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2 Reacting shock wave propagating in a periodic medium

When the upstream state is a non-zero constant, generalization of 1 is straightforward. One can investigate the effect of constant u_a on the stability of solutions of the reactive Burgers equation, and this is done in the present work. However, our main focus here is on the analysis of solutions when the upstream state is a spatially periodic function oscillating around some non-zero constant value (the average state). The central questions are then: How do the intrinsic oscillations of the solution caused by instability interact with oscillations due to the variable upstream state? Is there any amplification of oscillations (that is, resonance) due to this interaction?

To formulate the generalized model, we assume that $u_a = u_a(\xi)$ is a prescribed function of the laboratory-frame coordinate ξ. The shock-frame coordinate is then $x = \xi - \xi_s(t)$, where $\xi_s(t) = \int_0^t D(\tau)\, d\tau$ is the shock position in the laboratory frame, D is the shock speed. No steady state solution exists in this case as the state immediately in front of the shock changes with time as the shock propagates. For this reason, we choose to initialize the numerical simulations with a steady-state ZND solution that corresponds to a constant upstream state equal to the average of the given periodic state. The latter is taken as $u_a(\xi) = \bar{u}_a(1 + \sin k\xi)$, where $k = 2\pi/\lambda$ is the wavenumber, λ is the spatial wavelength of the nonuniformity. Thus, u_a oscillates around its average value $\bar{u}_a$ between 0 and $2\bar{u}_a$. In the computations below, the initial position of the lead shock is chosen such that u_a at the shock coincides with $\bar{u}_a$ so as to minimize any start-up transients.

The shock conditions now yield that $D = (u_s + u_a(\xi_s))/2$. If $\bar{u}_a$ denotes the uniform or the average upstream state, then we use the shock amplitude relative to the upstream, $u_{0s} - \bar{u}_a$, as a new scale for $u(x,t)$. Other scales are chosen as in [1] and we omit the details.

Then, the main equation becomes:

$$u_t + \left(\frac{u^2}{2} - Du\right)_x = f_0 \exp\left[-\left(\frac{x + ((1-\bar{u}_a)\, u_s)^{-\alpha}}{2\sqrt{\beta}}\right)^2\right] \tag{2}$$

with

$$D = \frac{1}{2}\left[u_s + \frac{\bar{u}_a}{1-\bar{u}_a}\left(1 + \sin(k\xi_s)\right)\right]. \tag{3}$$

Here parameters α and β are analogs of activation energy and the width of the heat release region, respectively, as explained in [3]. The factor f_0 is chosen such that the integral of the source term over $(-\infty, 0)$ is a constant. New parameters here are the amplitude $\bar{u}_a$ and wavelength k of the heterogeneity. The numerical solution of (2) requires not only the update of $u_s(t)$ with time, but also of the shock position $\xi_s(t)$ as the latter enters the expression 3 for D via the instantaneous state just upstream of the shock.

3 Computational results

For numerical solution of the problem we use the same shock-fitting approach as in [3] where the reader can find the details (only now the method is implemented in Python). Parameters $\beta = 0.1$ and $\alpha = 4.5$ are fixed in all our calculations while the upstream-state amplitude $\bar{u}_a$ and wavenumber k are varied. The numerical domain length is $L = 10$ units, and 200 grid points in the domain have been used in all of the calculations below, which provides sufficiently high accuracy for this shock-fitting method.

Figure 1 demonstrates how increasing the constant value of u_a (which is the same as $\bar{u}_a$ for this case) above zero leads to the stabilization of solutions. This effect allows us to explain some of the behavior seen for the case of a periodically and slowly varying upstream state as will be clear from the discussion below about Figure 2.

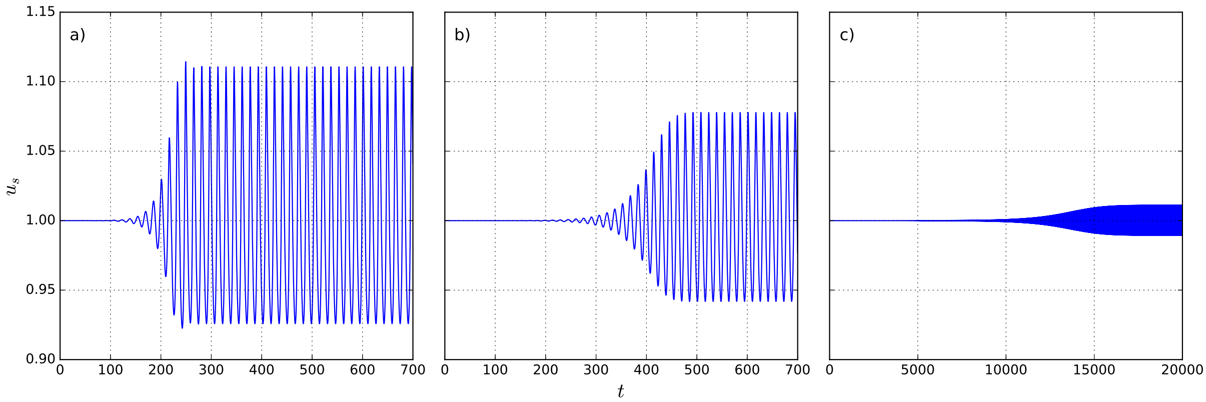

Figure 1: Stabilization of solutions when the uniform upstream state u_a increases: a) $u_a = 0$, b) $u_a = 0.05$, c) $u_a = 0.1$. Figures show the time series of the shock state $u_s(t)$.

Now consider the case of a periodic upstream state. We fix the values of α, β, and $\bar{u}_a$ and focus on the role played by the wavenumber k. Shown in Figure 2a) is the solution when $k = 0$, i.e. the uniform upstream state that we take as a reference. We see that the solution is a simple period-1 limit cycle with period $T_0 = 15.6$ and frequency $\nu_0 = 0.064$ (we will refer to this as the ”natural frequency“), as also seen in the power spectrum on the right side of Fig. 2a)).

In Figure 2b), we show the solution $u_s(t)$ at $k = 0.01$, i.e., when the upstream state varies slowly on the time scale of the detonation intrinsic oscillations. We observe that the shock wave generally follows the oscillations of u_a, stably at crests and with periodic bursts of instability when the shock wave passes through the troughs of u_a. This behavior is consistent with the observation in Figure 1, where the larger values of constant u_a correspond to more stable solutions. The wave packets arising in the trough phase are seen to correspond to period-doubled oscillations in this particular case. The power spectrum for this solution shows dominant peaks at $\nu_f = 0.001$ (due to forcing by the slowly varying upstream state) and at $\nu = 0.028, 0.056$, which arise due to intrinsic instability of the wave.

Next, we increase k making the upstream state more oscillatory. Figure 2c) shows the results obtained with $k = 0.1$. Even though the upstream state is still varying relatively slowly, we can now see stronger interactions between different modes yielding a more complex time series of u_s. The power spectrum shows strong peaks at $\nu_f = 0.009$ (coming from the forcing) and at $\nu = \nu_0 - \nu_f = 0.056$, $\nu = \nu_0 - 2\nu_f = 0.047$. Note that the natural mode is now relatively weak.

When we increase k to $k = 0.4$, shown in Figure 2d), we find that only one frequency and its harmonics are present, indicating the phenomenon of frequency locking at $\nu = 0.035$ which is approximately half of the natural frequency $\nu_0 = 0.065$.

Further increasing k leaves the frequency-locked oscillations for some range of k, however, more complex solutions do arise. For example, at $k = 0.75$, a period-doubled oscillation is seen, Figure 2e), with the dominant frequency now equal the natural frequency ν_0. At even larger k, we find that the solution returns to the same period-1 form as at $k = 0$ (Figure 2f) at $k = 3$), indicating that the solution no longer ”sees“ the rapid oscillations of the upstream.

In addition to the complexities of the mode interaction that are shown in Figure 2, we point out also the increased amplitude of the oscillations at the intermediate values of k which are indicative of a resonant

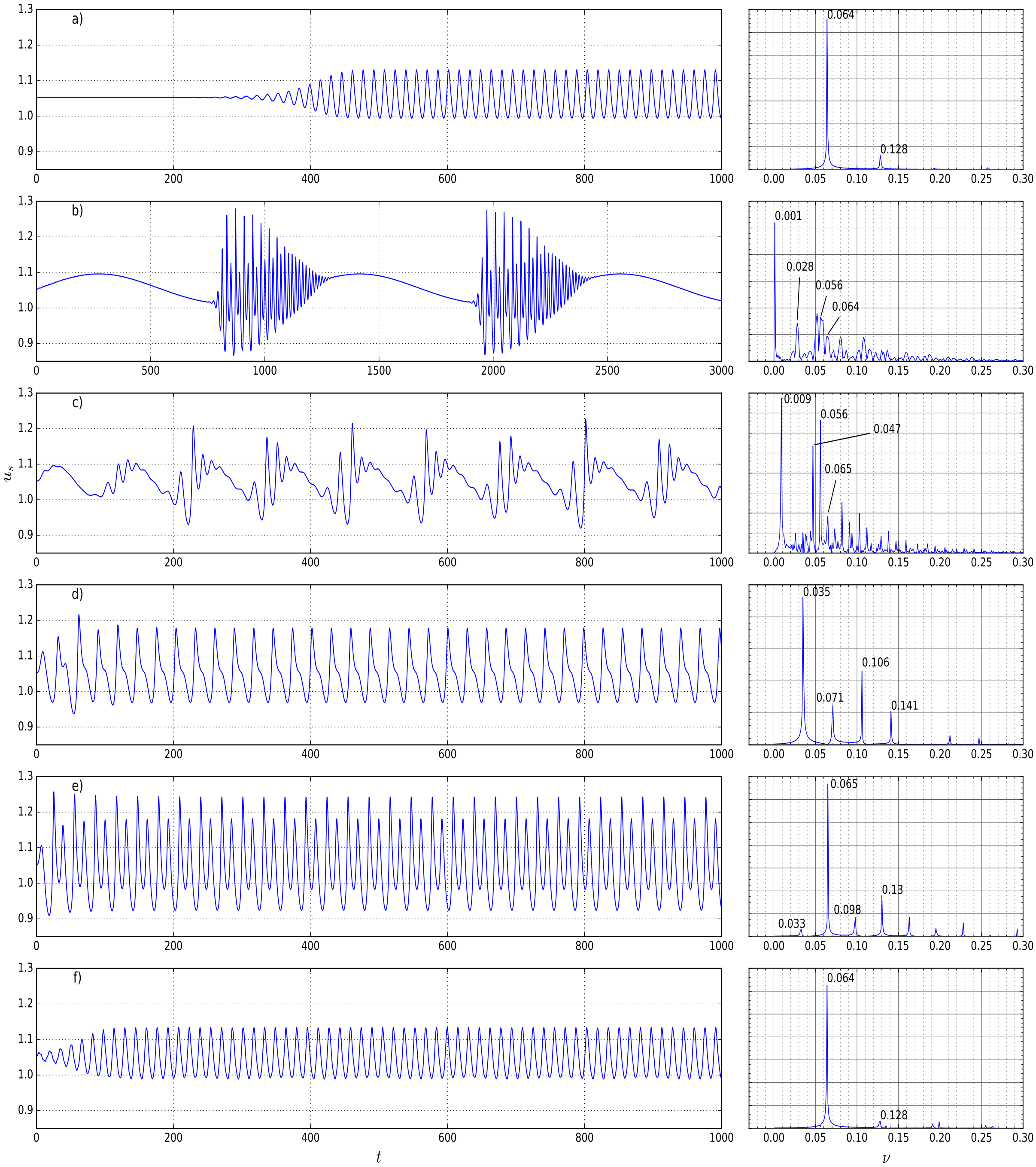

Figure 2: The effect of the wavelength of upstream-state variation on the solution: a) $k = 0$, b) $k = 0.01$, c) $k = 0.1$, d) $k = 0.4$, e) $k = 0.75$, f) $k = 3$. The left column shows the time series of $u_s(t)$ and the right column the corresponding power spectra.

interaction between the intrinsic oscillations and those caused by the periodic forcing.

4 Conclusions

In this work, we have used the analog Burgers model of detonation [3] in order to explore how detonation propagates in a medium with a periodically varying reactivity. Our main findings are: 1) that interactions with upstream-state periodicity lead to more complex nonlinear oscillations of the solution that includes existence of nonlinear wavepackets, period-doubling bifurcations, and frequency locking; 2) the oscillation amplitude is increased due to the interactions with heterogeneity indicating existence of a resonance.

A number of questions require further exploration, including that of existence in the reactive Burgers model of ”super-CJ” solutions (i.e. solutions with the average velocity higher than that in the respective homogeneous medium [4,5]), and the question about the role of more complex heterogeneities (discrete, multi-scale, random, etc.). An important extension of the present work is to the reactive Euler equations, and that will be pursued in follow-up work.

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