Stability of non-adiabatic shock waves

César Huete, Alberto Cuadra, and Marcos Vera
Fluid Mechanics Group, Escuela Politécnica Superior, Universidad Carlos III de Madrid
Madrid, Spain

1 Introduction

The stability of two-dimensional shocks has been continuously investigated since the pioneering works of D'Yakov [1] and Kontorovich [2]. It was found that, in certain conditions related to the Rankine-Hugoniot slope, the shock oscillates permanently with the corresponding generation of entropic and rotational perturbations downstream, as well as the emission of constant-amplitude sonic waves, the latter being commonly referred as spontaneous acoustic emission (SAE).

Motivated by the dominant role of shock waves in supersonic aerodynamics, and also in Inertial Confinement Fusion, there exists an exhaustive theoretical work on this subject [3–12], with the literature accumulated on the Richtmyer-Meshkov Instability deserving special recognition. By way of contrast, non-adiabatic waves have received considerably less attention, with regular detonations standing among any other type of supersonic reactive front due to its direct application in propulsion engines and safety issues [13–18]. Further examples of non-adiabatic supersonic fronts are found in shocks that can induce a phase change on its path [19, 20]. Hereafter the term non-adiabatic will comprise any type of energy gain or loss by the fluid particles across the shock, *e.g.,* reactive, dissociating, radiative or ionizing. The astrophysical context opens a wide branch of possibilities for non-adiabatic waveforms, as are thermonuclear detonations formed in type Ia supernovae [21, 22]. Other examples are found in core-collapse type II supernovae, where accretion shocks become endothermic due to the nuclear dissociation process [23–25]. A distinctive common feature of these two fronts is that energy variations may likely depend on thermodynamical absolute values.

As previously done by Bates [5] and Wouchuk [7], which predicted the possibility of DK-instability in gases governed by van der Waals forces, the present study shows that permanent oscillations and SAE can occur in exothermic strength-sensitive shocks. The possibility of highly-damped oscillations is also predicted in endothermic configurations, a regime that was previously associated to nonideal gases [6]. In this work, the problem formulation and its resolution are presented in a self-contained form. The methodology employs linear perturbation analysis in the thin-shock limit, where the perturbation wavelength $\lambda' = 2\pi/k'$ is much larger than the shock thickness ℓ, which comprises the precursor adiabatic shock and the following reacting layer. The model also assumes the isolated-shock setup and the steady-state condition in the background variables. Therefore, the instability threshold resulting from unsteadiness [26] and/or piston-driving effects [11] are not considered.

2 Problem formulation

It is defined an isolated shock whose relative speed with respect to the upstream flow is $u'_1 > a'_1$, where the speed of sound is modeled with the perfect gas equation of state $a'^2_1 = \gamma\, p'_1/\rho'_1$. The subscripts 1 and 2 will refer to upstream and downstream dimensional flow properties that include: velocity u', density ρ', pressure p', and enthalpy h'. The energy conservation equation then reads

$$\frac{\gamma}{\gamma-1}\frac{p'_1}{\rho'_1} + \frac{u'^2_1}{2} + \Delta h'_0 = \frac{\gamma}{\gamma-1}\frac{p'_2}{\rho'_2} + \frac{u'^2_2}{2}. \tag{1}$$

The factor $\Delta h'_0$ defines the energy per unit mass delivered to the fluid particles in case of considering exothermic effects ($\Delta h'_0 > 0$) or the energy per unit mass subtracted to the fluid particles when endothermic effects are considered ($\Delta h'_0 < 0$). This term is conveniently scaled with the upstream speed of sound to yield $\mathcal{H}_0 = \left(\gamma^2-1\right)\Delta h'_0/\left(2a'^2_1\right)$ as the dimensionless order-of-unity energy change.

When the energy variation across the shock depends on the shock intensity, the amount of energy delivered or taken from the fluid should be modeled according to the particular phenomenon taking place. It is found, for example, that accretion shocks formed in CCSNe are able to break heavy nuclei and the energy employed in the nuclear dissociation scales with the upstream energy flux [24]. Further examples of this kind can found in supersonic fronts that induce phase change [19]. Thermonuclear detonations may also depend on the shock intensity, as the amount of nuclei fused depends on density [21,22]. Although the scaling of heat release with the flow properties depends on the particular phenomenon considered, the upstream energy flux is a good candidate since it involves both the preshock state and the shock intensity. For this case, the dimensionless non-adiabaticity of the shock is modeled as $\mathcal{H}_0 = \varepsilon(\gamma+1)\left[2+(\gamma-1)\mathcal{M}_1^2\right]/4$, with ε being a constant parameter that represents the fraction of the incoming energy flux released to ($\varepsilon > 0$) or taken from ($\varepsilon < 0$) the fluid particles.

The variation of the different flow variables across the shock is readily obtained through the streamwise integration of the conservation equations, which yields

$$\mathcal{R}_s = \frac{\rho'_2}{\rho'_1} = \frac{u'_1}{u'_2} = \frac{(\gamma+1)\,\mathcal{M}_1^2}{(\gamma-\kappa)\,\mathcal{M}_1^2+1} \tag{2}$$

for the mass compression ratio. The Mach number relative to the postshock flow is

$$\mathcal{M}_2 = \frac{u'_2}{a'_2} = \left[\frac{(\gamma-\kappa)\,\mathcal{M}_1^2+1}{\gamma\mathcal{M}_1^2(1+\kappa)+1}\right]^{1/2}, \tag{3}$$

where the function $\kappa = \left[(1-\mathcal{M}_1^{-2})^2 - 4\mathcal{H}_0\mathcal{M}_1^{-2}\right]^{1/2}$ contains the dimensionless non-adiabatic parameter $\mathcal{H}_0$. The value of κ can be zero only in exothermic conditions $\mathcal{H}_0 > 0$. This condition is achieved at the so-called Chapman-Jouget regime, namely $\mathcal{M}_1^{-2} = 1+(\gamma+1)\varepsilon\left(1-\sqrt{1+\varepsilon^{-1}}\right)$. At this regime, the flow behind the reacting shock is sonic $\mathcal{M}_2 = 1$, thereby decoupling the shock from downstream influences. The corresponding Rakine-Hugoniot curve

$$\mathcal{P}_s = \frac{2\gamma\mathcal{R}_s\left(\mathcal{R}_s-1\right)(1-\varepsilon)-(\gamma-1)\left[\mathcal{R}_s^2\,(1-\varepsilon)-1\right]}{2\gamma\left(\mathcal{R}_s-1\right)-(\gamma-1)\left[\mathcal{R}_s^2\,(1-\varepsilon)-1\right]} \tag{4}$$

is written for convenience in the definition of the function Γ_s in (6).

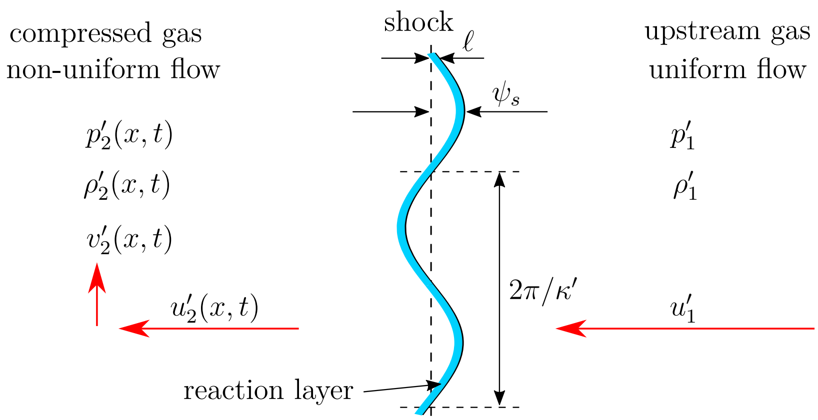

Figure 1: Sketch of the perturbed shock wave and the perturbation variables in the shock reference frame.

To study the evolution of the planar shock, it is assumed an initial ripple of the form $\psi_s(t=0) = \psi_{s0}\cos(k'y')$, with ψ_{s0} being the initial amplitude, k' its perturbation wavenumber, and y' the coordinate transverse to the shock direction, namely x'. The wavenumber is employed to scale the spatial and temporal variables, $x = k'x'$, $y = k'y'$, and $\tau = a'_2k't$, the dimensionless shock ripple is defined as $\xi_s = \psi_s/\psi_{s0}$, and the small factor $\psi_{s0}k'$ is used to scale the perturbation variables downstream, namely $\phi'(x,y,\tau) = \Phi'_2\left[1+\psi_{s0}k'\,\hat{\phi}(x,y,\tau)\right]$, with the function ϕ' denoting the dimensional variables p', ρ', u', and v'. Correspondingly, the associated dimensionless variable $\hat{\phi}$ defines order-of-unity functions. The base-flow factor Φ' takes the form $\rho'_2a'^2_2$, ρ'_2, and a'_2, for pressure, density and velocity, respectively.

The formulation of the asymptotic problem calls for information about the burnt-gas governing equations, i.e., the linear Euler equations, and the boundary conditions. One is given by the values at the shock $\hat{\phi}(x=\mathcal{M}_2\tau) = \hat{\phi}_s$, which are determined by linearized Rankine-Hugoniot equations

$$\dot{\xi}_s(\tau) = \frac{\mathcal{R}_s}{\mathcal{R}_s-1}\frac{1-\Gamma_s}{2\mathcal{M}_2}\hat{p}_s(\tau), \tag{5a}$$

$$\hat{u}_s(\tau) = \frac{1+\Gamma_s}{2\mathcal{M}_2}\hat{p}_s(\tau), \tag{5b}$$

$$\hat{\rho}_s(\tau) = \frac{\Gamma_s}{\mathcal{M}_2^2}\hat{p}_s(\tau), \tag{5c}$$

$$\hat{v}_s(\tau) = \mathcal{M}_2(\mathcal{R}_s-1)\frac{\partial\xi_s}{\partial y}(\tau). \tag{5d}$$

The function

$$\Gamma_s = u'^2_2\left(\frac{\partial p'_2}{\partial\rho'_2}\right)^{-1} = \frac{\gamma\mathcal{M}_1^2}{\mathcal{R}_s^2}\left(\frac{\partial\mathcal{P}_s}{\partial\mathcal{R}_s}\right)^{-1} \tag{6}$$

relates to the celebrated D'Yakov parameter [1, 2] and it reduces to $\mathcal{M}_1^{-2}$ in the adiabatic limit. The other boundary condition is provided by the isolated-shock assumption, which translates into not considering the effect of the acoustic waves reaching the shock from behind. For this condition to be true, the shock must be sufficiently far from driving conditions, which always fulfills when $\mathcal{M}_1/\mathcal{M}_{\rm cj}\sim 1$. Besides, the linear theory and the thin-shock assumptions set the following limits: $\ell \ll \psi_{s0} \ll k'^{-1}$, see Fig.1.

3 Asymptotic shock dynamics

For the shock to spontaneously, and constantly, radiate acoustic waves downstream when $\tau \gg 1$, it must oscillate with a non-decaying amplitude with sufficiently high frequency, so that the acoustic wavenumber vector points backwards in the shock reference frame. This condition is deduced anticipating that sonic waves are functions of $(\omega_a \tau - k_a x)$. The dimensionless frequency ω_a and wavenumber k_a are obtained from the adiabatic dispersion relation $\omega_a^2 = k_a^2 + 1$ along with the compatibility condition at the shock $\omega = \omega_a - \mathcal{M}_2 k_a$. It is readily seen that constant-amplitude perturbations take place for $\omega \geq \sqrt{1-\mathcal{M}_2^2}$.

The most relevant information relative to the asymptotic behavior can be inferred from the Laplace Transform of the shock ripple amplitude [6, 7], namely

$$\int_0^\infty \xi_s(r) e^{-sr} \mathrm{d}r = \frac{\sqrt{s^2+1} + \sigma_b s}{s\sqrt{s^2+1} + \sigma_b s^2 + \sigma_c} \tag{7}$$

applied over the variable $r = \tau\sqrt{\mathcal{M}_2^2 - 1}$, with the auxiliary factors defined as

$$\sigma_b = \frac{1+\Gamma_s}{2\mathcal{M}_2} \quad \text{and} \quad \sigma_c = \frac{\mathcal{R}_s \mathcal{M}_2}{1-\mathcal{M}_2^2} \frac{1-\Gamma_s}{2}. \tag{8}$$

A complete derivation of the type of modes in (7) can be found in [27]. The condition that sets the limits for stable oscillations is $s = \pm \mathrm{i}$, which occurs for $\sigma_b = \sigma_c$, which, in the temporal domain, translates into damped oscillations whose asymptotic decaying rate is proportional to $\tau^{-1/2}$. Permanent oscillations at the shock are found to happen when $\sigma_b < \sigma_c$, that is, when the imaginary poles in (7) lie outside the branch cut. The asymptotic oscillation frequency is $\omega = \Omega\sqrt{1-\mathcal{M}_2^2}$, with

$$\Omega^2 = \frac{2\sigma_b\sigma_c - 1}{2\left(\sigma_b^2 - 1\right)} \left[1 - \frac{\sqrt{4\sigma_c\left(\sigma_c - \sigma_b\right) + 1}}{2\sigma_b\sigma_c - 1}\right]. \tag{9}$$

As $\Omega \sim 1$ for $\sigma_b < \sigma_c$, the condition $\omega \geq \sqrt{1-\mathcal{M}_2^2}$ is always satisfied within this regime. It implies that permanent oscillations at the shock wave that moves free from external perturbations exhibit SAE. The DK-stability limit depends on the local slope of the Rankine-Hugoniot curve in the postshock state. This is commonly presented as a function of the D'Yakov–Kontorovich parameter [1, 2]

$$\varphi_{\mathrm{rad}} = \frac{\mathcal{M}_2^2\left(\mathcal{R}_s + 1\right) - 1}{\mathcal{M}_2^2\left(\mathcal{R}_s - 1\right) + 1}, \tag{10}$$

which separates the acoustically radiating from the non-radiating condition, with $\Gamma_s = \varphi_{\mathrm{rad}}$ being a condition similar to that $\sigma_b = \sigma_c$ derived above. For $\sigma_b > \sigma_c$ the shock front oscillates towards the asymptotic planar solution with an amplitude that decays in time with the power law $\tau^{-3/2}$. There exist, however, two distinguished scenarios: when $\sigma_b > \sigma_c + 1/(4\sigma_c)$ (or $\Lambda < 0$ as defined in [6]), the approach towards the asymptotic decay rate occurs faster than that occurring in regular conditions, namely $\sigma_c + 1/(4\sigma_c) > \sigma_b > \sigma_c$. Regular conditions are associated to finite-strength shocks moving adiabatically in perfect gases. The limit that distinguishes fast or regular damping can be also expressed as $\Gamma_s = \varphi_{\mathrm{dam}}$ [27], with

$$\varphi_{\mathrm{dam}} = \frac{\mathcal{R}_s\mathcal{M}_2^2 - \left(1-\mathcal{M}_2^2\right)^{3/2}\sqrt{1-\mathcal{R}_s^{-1}}}{\mathcal{R}_s\mathcal{M}_2^2 + 1 - \mathcal{M}_2^2}. \tag{11}$$

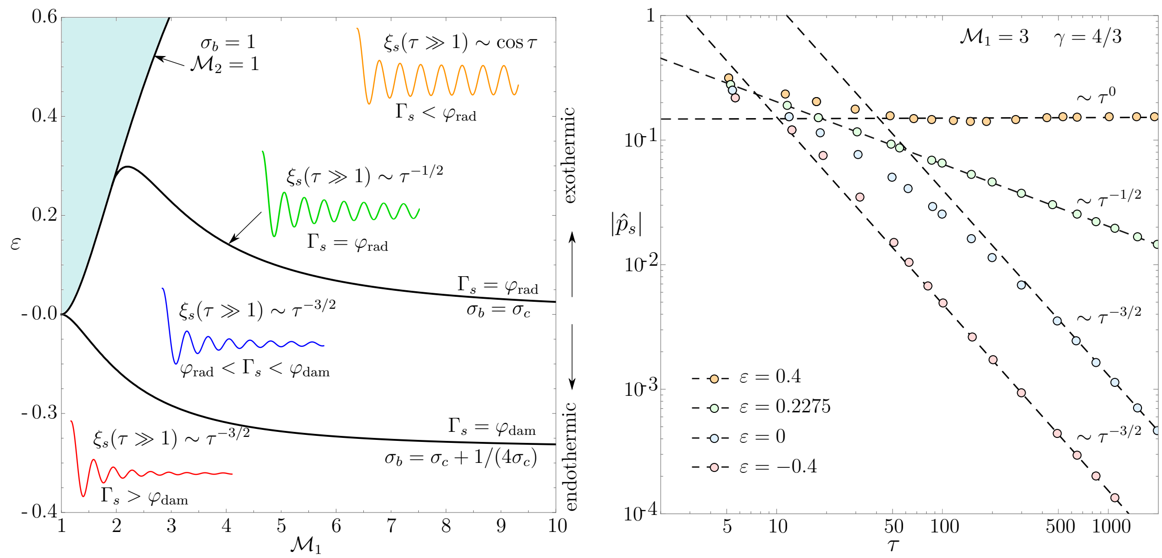

Figure 2: Left: asymptotic regimes as a function of $\mathcal{M}_1$ and the corresponding non-adiabatic factors q and ε, for $\gamma = 4/3$. The blue-shaded region corresponds to values of $\mathcal{M}_1 < \mathcal{M}_{\rm cj}$ ($\mathcal{M}_2 < 1$). Right: Shock pressure peaks for four $\varepsilon = 0.4$, $\varepsilon = 0$, $\varepsilon = -0.2275$, and $\varepsilon = -0.4$ (circles).

The aforementioned regimes are computed for $\gamma = 4/3$ in Fig. 2 (left). The blue-shaded zones correspond to values of $\mathcal{M}_1 < \mathcal{M}_{\rm cj}$ that lead supersonic flows downstream. For sufficiently weak shocks, the effect of the shock intensity in the energy change is negligible and the constant contribution dominates the function $\mathcal{H}_0 \sim \varepsilon(\gamma+1)/2 \sim \varepsilon$. In this weak-shock limit, there is no room for stable oscillations as the limiting curve $\sigma_b = \sigma_c$ meets the boundary $\mathcal{M}_2 = 1$. For finite-shock intensities, of the order of $\mathcal{M}_1^2 \sim 2/(\gamma-1)$, the strength-sensitive contribution in $\mathcal{H}_0$ is no longer negligible and the curve $\sigma_b = \sigma_c$ detaches from the boundary $\mathcal{M}_2 = 1$, then enabling the possibility of permanent oscillations for sufficiently exothermic shocks. In the strong shock limit, $\mathcal{M}_1^2 \gg 1$, the constant contribution in $\mathcal{H}_0$ can be neglected so that $\mathcal{H}_0 \sim \varepsilon(\gamma^2-1)\mathcal{M}_1^2/4$. The curve $\sigma_b = \sigma_c$ approaches the adiabatic limit from the exothermic side. It implies that strong shocks will lie in the permanent-oscillating regime when heat release is sensitive to absolute properties like temperature or pressure, even for fairly small sensitivities. As previously reported [14, 27], planar reactive shocks whose overall heat released is invariant to the shock strength are stable to long-wavelength perturbations. Fig. 2 (right) shows the computations of the shock pressure perturbations evaluated through Bessel functions (solution of the two-dimensional wave equation with periodic symmetry in y) after solving the initial-value problem. Small changes in γ does not change the qualitative picture.

4 Conclusions

It is shown that the D'Yakov-Kontorovich instability is not uniquely restricted to shocks moving through nonideal gases. Non-adiabatic effects can make the shock behave in well-distinguished regimes that includes the stable oscillatory mode. An expected finding is that endothermic effects tend to attenuate the shock oscillations. For sufficiently endothermic shocks, the shock vibrations exhibit a higher damping towards the asymptotic decay rate $\tau^{-3/2}$. On the other side, exothermic transformations may induce constant-amplitude vibrations when the amount of heat released is positively correlated to the shock strength, a distinctive feature of reactions that are density, pressure or temperature dependent. For sufficiently strong shocks, even small sensitivities have been found to switch the shock dynamics to the permanent-oscillating regime and, consequently, to radiate acoustic waves. For exothermic shocks, results may be applicable to detonations whose net amount of heat release depends on the postshock state as long as the reaction zone does not exhibit an unstable behavior. Otherwise, the inner structure must be taken into account in the computation of the downstream flow-field. For endothermic scenarios, admitting the simplifications made in modeling the gas, the finding may be suitable to nuclear dissociating, radiative or ionizing shocks. Whatever the scenario considered, the limits defining the distinguished regimes should be adapted to include the particular non-adiabaticity model, since others may not strictly scale with the upstream energy flux. Moreover, effects neglected in this work, like are plausible base-flow unsteadiness and downstream coupling, should be also considered in more realistic scenarios.

Acknowledgments

This work has been supported by the Ministry of Science (ENE2015-65852-C2-1-R) and Fundación Iberdrola España (BINV-hBbhOeJQ).

References

[1] S.P. D'Yakov, The stability of shockwaves: Investigation of the problem of stability of shock waves in arbitrary media, Zh. Eksp. Teor. Fiz. **27**, 288–295 (1954).

[2] V.M. Kontorovich, On the stability of shock waves, Zh. Eksp. Teor. Fiz. **33**, 1525–1526 (1957).

[3] N.C. Freeman, A Theory of the Stability of Plane Shock Waves, Phil. Trans. R. Soc. A. **228** 341-362 (1955).

[4] G.R. Fowles and G.W. Swan, Stability of plane shock waves, Phys. Rev. Lett. **30**, 1023–1025 (1973).

[5] J.W. Bates, The D'Yakov–Kontorovich instability of shock waves in real gases, Phys. Rev. Lett. **84**, 1180 (2000).

[6] J.W. Bates and D.C. Montgomery, Initial-value-problem solution for isolated rippled shock fronts in arbitrary fluid media, Phys. Rev. E **69**, 056313 (2004).

[7] J.G. Wouchuk and J. López Cabada, Spontaneous acoustic emission of a corrugated shock wave in the presence of a reflecting surface, Phys. Rev. E **70**, 046303 (2004).

[8] J.W. Bates, Instability of isolated planar shock waves, Phys. Fluids **19**, 094102 (2007).

[9] A.V. Konyukhov, A.P., Likhachev, V.E. Fortov, K.V. Khishchenko, S.I. Anisimov, A.M. Oparin, I.V. Lomonosov On the neutral stability of a shock wave in real media *JETP Lett.* **90**, 18–24 (2009).

[10] J.W. Bates, On the theory of a shock wave driven by a corrugated piston in a non–ideal fluid, J. Fluid Mech. **691**, 146–164 (2012).

[11] J.W. Bates, Theory of the corrugation instability of a piston-driven shock wave, Phys. Rev. E **91** 013014 (2015).

[12] N. Wetta, J.C. Pain, and O. Heuzé, D'Yakov-Kontorovitch instability of shock waves in hot plasmas, *Phys. Rev. E* **98** 033205 (2018).

[13] J.J. Erpenbeck, Stability of idealized one–reaction detonations, Phys. Fluids **7**, 684–696 (1964).

[14] A.J. Majda & R.R. Rosales, A theory for spontaneous Mach stem formation in reacting shock fronts, SIAM J. Appl. Math. **21**, 1310–1334 (1983).

[15] M. Short and D.S. Stewart, The multi-dimensional stability of weak–heat–release detonations, J. Fluid Mech. **382**, 109–135 (1999).

[16] R. Daou and P. Clavin, Instability threshold of gaseous detonations, J. Fluid Mech. **482**, 181–206 (2003).

[17] P. Clavin and F.A. Williams, Analytical studies of the dynamics of gaseous detonations, Phil. Trans. R. Soc. A. **13** 597-624 (2012).

[18] C. Huete, T. Jin, D. Martnez-Ruiz, and K. Luo, Interaction of a planar reacting shock wave with an isotropic turbulent vorticity field, Phys. Rev. E **96**, 053104 (2017).

[19] A. Guha, A unified theory of aerodynamic and condensation shock waves in vapor–droplet flows with or without a carrier gas, Phys. Fluids **6**, 1893–1913 (1994).

[20] D. Vimercati, G. Gori, and A. Guardone, Non-ideal oblique shock waves, J. Fluid Mech. **847**, 266–285 (2018).

[21] A.M. Khokhlov, The structure of detonation waves in supernovae, Mon. Notices Royal Astron. Soc. **239**, 785–808 (1988).

[22] V.N. Gamezo, A.M. Khokhlov, and E.S. Oran, Deflagrations and detonations in thermonuclear supernovae, Phys. Rev. Lett. **92**, 211102 (2004).

[23] T. Foglizzo, P. Galletti, L. Scheck, and H.-T. Janka, Instability of a stalled accretion shock: evidence for the advective-acoustic cycle. Astrophys. J.,**654** 1006–1021 (2007).

[24] R. Fernández and C. Thompson, Stability of a spherical accretion shock with nuclear dissociation, Astrophys. J. **697**, 1827–1841 (2009).

[25] C. Huete, E. Abdikamalov, and D. Radice, The impact of vorticity waves on the shock dynamics in core-collapse supernovae, Mon. Notices Royal Astron. Soc. **475**, 3305–3323 (2018).

[26] E.T. Vishniac and D. Ryu, On the stability of decelerating shocks." (1989). Astrophys. J., **337**, 917–926 (1989).

[27] P. Clavin and G. Searby, *Combustion waves and fronts in flows: flames, shocks, detonations, ablation fronts and explosion of stars*, (Cambridge University Press, 2016).

[28] P. Clavin, Nonlinear analysis of shock–vortex interaction: Mach stem formation, J. Fluid Mech. **721**, 324–339 (2013).