Stationary Premixed Flames in Narrow Tubes with External Heat Transfer

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1 Introduction

Because of their ability to burn very lean mixtures, there has been significant interest in superadiabatic flames recently. Significant work, experimental, numerical and analytical, is reviewed in [1]. Here, a steady flame occurs in a long narrow tube exposed to a known temperature on its outside wall. The formulation is similar to [1–6], but focus is on exact analytical solutions to the one-dimensional problem using a convection model for heat transfer on the tube walls. The technique [1] is similar to ref. [7], but applied to the full tube instead of half, and boundary conditions allow for superadiabaticity. The work differs from [2–5] in that a closed form solution is obtained, restricted to a fixed flame location. Advantages of an analytical solution are (1) that physics of the solution are clarified, such as, for instance, thicknesses of the various zones, and (2) that it is more reliable and efficient than numerical solutions.

2 Formulation

Species and energy conservation are:

$$m\frac{d\bar{Y}_{F}}{d\bar{x}} = \bar{\rho}D_{F}\frac{d^{2}\bar{Y}_{F}}{d\bar{x}^{2}} - \frac{B\bar{\rho}^{2}\bar{Y}_{O}\bar{Y}_{F}e^{-E_{a}/R_{u}\bar{T}_{g}}}{M_{O}M_{F}}, \quad m\frac{d\bar{Y}_{O}}{d\bar{x}} = \bar{\rho}D_{O}\frac{d^{2}\bar{Y}_{O}}{d\bar{x}^{2}} - \phi\frac{B\bar{\rho}^{2}\bar{Y}_{O}\bar{Y}_{F}e^{-E_{a}/R_{u}\bar{T}_{g}}}{M_{O}M_{F}}$$
(1)

$$mc_{p}\frac{d\bar{T}_{g}}{d\bar{x}} = k_{g}\frac{d^{2}\bar{T}_{g}}{d\bar{x}^{2}} + \frac{\bar{h}_{v}}{A_{g}}(\bar{T}_{s} - \bar{T}_{g}) + \bar{Q}\frac{B\bar{\rho}^{2}\bar{Y}_{O}\bar{Y}_{F} \ e^{-E_{a}/R_{u}\bar{T}_{g}}}{M_{O}M_{F}}$$
(2)

$$0 = k_s \frac{d^2 \bar{T}_s}{d\bar{x}^2} - \frac{\bar{h}_v}{A_s} (\bar{T}_s - \bar{T}_g) - \frac{\bar{h}_w}{A_w} (\bar{T}_s - \bar{T}_w),$$
(3)

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With fluid temperature boundary condition \overline{T}_0 at left infinity, temperature outside the wall \overline{T}_w and species BCs fully unburnt, \overline{Y}_{F0} and \overline{Y}_{O0} at $-\infty$, and fully burnt at $+\infty$. Scaling: temperatures by $\overline{T}_1 - \overline{T}_0 = \overline{Y}_{F0}\overline{Q}/c_P$ and mass fractions by \overline{Y}_{F0} and \overline{Y}_{O0} , and also:

$$x = \sqrt{\frac{\bar{h}_v}{A_s k_s}} \bar{x}, \quad J = \frac{A_s k_s}{A_g k_g}, \quad \Gamma = \frac{\sqrt{k_s A_s} m c_p}{k_g \sqrt{\bar{h}_v}}, \quad N_o = \frac{\bar{h}_w}{h_v} \frac{A_s}{A_w}$$
(4)

$$\frac{dY_F}{dx} = \frac{1}{\Gamma Le_F} \frac{d^2 Y_F}{dx^2} - \omega, \quad \frac{dY_O}{dx} = \frac{1}{\Gamma Le_O} \frac{d^2 Y_O}{dx^2} - \phi\omega$$
(5)

$$\frac{dT_g}{dx} = \frac{1}{\Gamma} \frac{d^2 T_g}{dx^2} + \frac{J}{\Gamma} (T_s - T_g) + \omega, \quad 0 = \frac{d^2 T_s}{dx^2} - (T_s - T_g) - N_o (T_s - T_w)$$
(6)

with Le_F and Le_O the fuel and oxidant Lewis numbers, and

$$\omega = \frac{\Gamma k_g}{m^2 c_p} \frac{B\bar{\rho}^2 \bar{Y}_{O0} Y_O Y_F \ e^{-E_a/R_u \bar{T}_g}}{M_O M_F} \tag{7}$$

For single step high activation energy kinetics, the reaction zone is reduced to a thin layer at x = 0 where due to heat release, the dimensionless fluid temperature gradient experiences a discontinuity $-\Gamma$, as determined by integration over the thin layer [6].

3 Solution

Eliminating T_q ,

$$\frac{d^4 T_s}{dx^4} - \Gamma \frac{d^3 T_s}{dx^3} - (1 + J + N_o) \frac{d^2 T_s}{dx^2} + (1 + N_o) \Gamma \frac{dT_s}{dx} + J N_o T_s = -N_o \left(\frac{d^2 T_w}{dx^2} - \Gamma \frac{dT_w}{dx} - J T_w\right)$$
(8)

The characteristic equation is:

$$r^{4} - \Gamma r^{3} - (1 + J + N_{o})r^{2} + (1 + N_{o})\Gamma r + JN_{o} = 0$$
(9)

The inverse of each of the roots of the characteristic equation determines the thickness of the various relevant zones. Closed form solutions can be obtained in two interesting cases: for $N_o = \Gamma^2$, and for the limit cases of a small J. Using perturbation, solutions can also be obtained in the neighborhood of these limits. The case where $N_o = \Gamma^2$ is only a special case in that, after factoring $r - \Gamma$, in the resulting cubic equation, the term in r^2 is zero. Otherwise, the solution would remain the same as below for the general case, if factoring $r - r_0$, where the root r_0 would be obtained numerically. Thus the only real particularity of the case solved below is that it can entirely be solved analytically. Admittedly $N_o = \Gamma^2$ may not be realistic for large Γ . The current dimensionless formulation is not appropriate with the limit cases of Γ either large or small.

4 Γ and J large, with ratio of order unity

Then, calling $G = J/\Gamma$, the problem is reduced to third order, hence an analytical solution:

$$\frac{d^3T_s}{dx^3} + G\frac{d^2T_s}{dx^2} - (1+N_o)\frac{dT_s}{dx} - N_oT_s = -N_o\left(\frac{dT_w}{dx} + GT_w\right)$$
(10)

The characteristic equation is:

$$r^{3} + \frac{J}{\Gamma}r^{2} - (1+N_{o})r - N_{o} = 0$$
(11)

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5 N_o close to Γ^2

For $N_o = \Gamma^2$, one root is $r_0 = \Gamma$. The other ones are solutions of

$$r^{3} - (1 + \Gamma^{2} + J)r - J\Gamma = 0$$
(12)

Using the Tartaglia method, these roots are

$$r_i = 2\bar{r}^{1/3} \cos \frac{\varphi + 2(i-1)\pi}{3}$$
(13)

with

$$\bar{r} = \left(\frac{1+\Gamma^2+J}{3}\right)^{3/2}, \quad \cos\varphi = \frac{J\Gamma}{2\bar{r}} = \frac{3^{3/2}}{2}\frac{J\Gamma}{(1+\Gamma^2+J)^{3/2}}$$
(14)

As a function of J, analyzing its derivative, $\cos \varphi$ is maximum for $J = 2(1 + \Gamma)^2$ with value $\cos \varphi_{max} = \Gamma/\sqrt{1 + \Gamma^2}$. Thus $0 \le \varphi/3 \le \pi/6$, as also shown in Fig. 1. Since $|\cos \varphi| \le 1$ unconditionally, all three roots are real. In view of the value of $\varphi/3$, one root r_1 is positive while $r_{2,3}$ are negative. Their value is shown in Fig. 2.



Figure 1: $\varphi/3$ vs. Γ , for J from 0.01 to 100.

Near root r, from Eq. (9),

$$\frac{dr}{dN_o} = \frac{r^2 - \Gamma r - J}{4r^3 - 3\Gamma r^2 - 2(1 + J + N_o)r + (1 + N_o)\Gamma}$$
(15)

For $r = \Gamma$, with $N_o = \Gamma^2$,

$$\frac{dr}{dN_o} = \frac{J}{(1+2J)\Gamma} \tag{16}$$

For the other roots, still with $N_o = \Gamma^2$, and using the characteristic equation, one can show that the denominator also remains nonzero, so that perturbations can be used for N_o close to Γ^2 .

For a constant T_w , accounting for BCs at $\pm \infty$, the solution has the form

$$T_s = T_0 + C_0 \exp \Gamma x + C_1 \exp r_1 x \text{ for } x < 0, \quad T_s = T_w + C_2 \exp r_2 x + C_3 \exp r_3 x \text{ for } x > 0$$
(17)

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Figure 2: Roots, r_1 left, $-r_2$ middle, $-r_3$ right, vs. Γ , for J from 0.01 to 100.

Continuity of T_s and its derivative across yields $T_0 + C_0 + C_1 = T_w + C_2 + C_3$ and $\Gamma C_0 + r_1 C_1 = r_2 C_2 + r_3 C_3$. Continuity of T_g , and the jump in dT_g/dx at the flame result in:

$$C_0\Gamma^2 + C_1r_1^2 = C_2r_2^2 + C_3r_3^2, \quad \Gamma^3C_0 + C_1r_1^3 - C_2r_2^3 - C_3r_3^3 = -\Gamma$$
(18)

with solution

$$C_0 = \frac{-r_1 r_2 r_3 (T_w - T_0) - \Gamma}{(\Gamma - r_1)(\Gamma - r_2)(\Gamma - r_3)}$$
(19)

$$C_{1} = \frac{[r_{2}r_{3}(T_{w} - T_{0}) + 1]\Gamma}{(\Gamma - r_{1})(r_{1} - r_{2})(r_{1} - r_{3})}, \quad C_{2} = \frac{-[r_{1}r_{3}(T_{w} - T_{0}) - 1]\Gamma}{(\Gamma - r_{2})(r_{2} - r_{1})(r_{2} - r_{3})}, \quad C_{3} = \frac{-[r_{1}r_{2}(T_{w} - T_{0}) - 1]\Gamma}{(\Gamma - r_{3})(r_{3} - r_{2})(r_{3} - r_{1})}$$
(20)

For the fluid temperature, a similar expression is obtained but with coefficients B_i :

$$B_0 = (1 + 2\Gamma^2)C_0, \quad B_1 = (1 + \Gamma^2 + r_1^2)C_1, \quad B_2 = (1 + \Gamma^2 + r_2^2)C_2, \quad B_3 = (1 + \Gamma^2 + r_3^2)C_3$$
(21)

Figures 3 to 5 show fluid and tube temperatures for J from 0.01 to 100., respectively for $\Gamma = 0.5$, 1.0 and 2.0, and for the common case where $T_w = T_0$. In that range the assumption $N = \Gamma^2$ remains reasonably realistic.

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Figure 3: Left: fluid temperature; Right: wall temperature. $\Gamma = 0.5$, J from 0.01 to 100.



Figure 4: Left: fluid temperature; Right: wall temperature. $\Gamma = 1.0$, J from 0.01 to 100.

6 Arbitrary No

For arbitrary N_0 one needs to solve numerically for the first root. Alternatively, it is much easier to pick a root value, and calculate the corresponding value of N_o . At least for higher Γ , the value above corresponds to unrealistically high heat transfer coefficient with the wall. So focusing to lower value, we pick $r_0 = \alpha \Gamma$ and focus on relatively small α . Replacing in the characteristic equation, and solving for N_o ,

$$N_{o} = \frac{\Gamma^{2} \alpha [(\alpha^{2} \Gamma^{2} - 1)(\alpha - 1) - J\alpha]}{\Gamma^{2} \alpha (\alpha - 1) - J}$$
(22)

Then for the other three roots, a solution using the Tartaglia method is readily implemented, but the third order equation now includes a term in r_2 . Thus the solution will be obtained using the very same procedure as in Fachini and Bauwens [1]. It is easier to use r_0 as the actual independent variable, for which a closed form value of N_o is readily computed, but to present results using N_o as the independent variable. Results will be presented for realistic values of N_o for higher values of Γ and as above, for a range of values of J.



Figure 5: Left: fluid temperature; Right: wall temperature. $\Gamma = 2.0$, J from 0.01 to 100.

7 Summary

A closed form entirely analytical solution was obtained for the problem of one-dimensional premixed flame in a long tube with external heat transfer to the environment, using a physical model in which an empirical heat transfer coefficient describes the heat exchange on both sides of the wall. A few results were presented, illustrating the strength of the approach. The key advantage of an analytical solution is that it readily allows for systematic investigation of the parameter space, or at least of large subsets. The analytical solution is much more cost-effective than numerical solutions which entail being redone for any individual cases.

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