Spectral and nonlinear stability of viscous strong and weak detonation waves in Majda's qualitative model

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1 Introduction: The Majda model and the Evans function

Since its introduction in the early 1980s, Majda's simplified "qualitative" model for detonation [9] has played an important role in the mathematical literature as test-bed for both the development of mathematical theory and computational techniques. Roughly, this model is a 2×2 system consisting of a Burgers equation coupled to a chemical kinetics equation. For example, Majda (together with Colella & Roytburd) used this model as a key diagnostic tool in the development of fractional-step computational schemes for the Navier–Stokes equations of compressible reacting fluids [2]. Further examples illustrating this model's varying roles in the literature include an analysis of the mathematical well-posedness of the model [6] and a rigorous study of the infinite-reaction-rate limit [7]. These highlighted examples serve to give an indication of the continuing mathematical interest in this model; the recent paper [4] gives a more detailed survey of the substantial mathematical literature dealing with the various incarnations of this kind of simplified combustion model.

Of course, one hopes that such simplified models will, on the one hand, capture many of the phenomena of the much more complicated reactive Euler and Navier–Stokes equations governing physical detonations, but will, on the other hand, eliminate enough of the technical complexities so as to be mathematically tractable. Here, our focus is on strong and weak detonations—particular traveling-wave solutions of Majda's model—and especially their stability. We address the stability at the spectral (linear) level, and at the *nonlinear* level. Our results show that, while Majda's model does a good job of reproducing the steady-state "wave" behavior, these waves do not seem to have the same dynamical/stability behavior as their counterparts in the associated physical systems. Indeed, we find all tested waves to be stable, and this is at odds with the known and expected behavior of these waves in the physical system.

Our analysis is based on a collection of techniques associated with the *Evans function*, denoted by E. The Evans function is a spectral determinant associated with a given traveling-wave solution; it is an analytic function whose zeros in the unstable complex plane correspond to instabilities (unstable eigenvalues) for that particular wave. Briefly, for strong- and weak-detonation solutions of the Majda model, we are able to show that the corresponding Evans functions do not vanish on the unstable complex half

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plane using a two-step strategy. For each wave, the first step is to show that there are no unstable unstable zeros of E outside of circle of some positive radius, say R. This we do analytically by means of an energy estimate. The second step is to eliminate the possibility of unstable zeroes inside

$$\{z \in \mathbb{C} \ : \ |z| \leq R \, , \mathrm{Re}\, z \geq 0\} \, .$$

We do this by numerically approximating E. However, our Evans-based analysis allows us to say more. Indeed, a technical analysis yields pointwise bounds on the Green function for the linear evolution equation for the perturbation [8], and these bounds in turn are sufficient to close a nonlinear argument that establishes the following proposition.

Proposition 1 (Lyng-Raoofi-Texier-Zumbrun [8]) *Consider the Majda system* (2)*. Under the Evans-function condition*

$$E(\cdot) \text{ has precisely one zero in } \{\mathrm{Re}\, \lambda \geq 0\} \text{ (necessarily at } \lambda = 0\text{)}, \tag{$\star$}$$

a strong or weak detonation wave of (2) *below is* $\hat{L}^\infty \to L^p$ *nonlinearly phase-asymptotically orbitally stable, for* $p > 1$*. Here,*

$$\hat{L}^\infty(\mathbb{R}) := \{f \in \mathscr{S}'(\mathbb{R}) \ : \ (1 + |\cdot|)^{3/2} f(\cdot) \in L^\infty(\mathbb{R})\}. \tag{1}$$

Remark 1 *We recall that if* X *and* Y *are Banach spaces, a standing-wave solution,* $\hat{U} = \hat{U}(x)$*, is said to be* $X \to Y$ nonlinearly orbitally stable *if, given initial data* U_0 *close in* X *to the wave, there is a phase shift* $\delta = \delta(t)$ *such that the perturbed solution emanating from data* U_0*, here denoted by* U*, satisfies*

$$\lim_{t\to+\infty} \|\hat{U}(x - \delta(t)) - U(\cdot, t)\|_Y = 0 \, .$$

That is, the perturbed solution approaches the δ*-shifted profile in* Y *as* $t \to \infty$*. If, in addition, the phase shift* $\delta(t)$ *converges to a limiting value* $\delta(+\infty)$*, we say that the wave is* nonlinearly phase-asymptotically orbitally stable.

The upshot is that the spectral stability calculations described above—showing that E has no zeros (eigenvalues) in the unstable complex half plane—allow us to use Proposition 1 to conclude nonlinear stability for both weak and strong detonation-wave solutions of Majda's model [3, 4]. In the following sections, we outline some of the key components of the analysis and associated computations.

2 The model and traveling waves

The model takes the form

$$u_t + \left(\frac{u^2}{2}\right)_x = Bu_{xx} + qk\phi(u)z \, , \tag{2a}$$

$$z_t = Dz_{xx} - k\phi(u)z \, . \tag{2b}$$

Here, the real-valued unknown u is an lumped variable to be thought of as representing variously density, velocity, and/or temperature. The unknown $z \in [0, 1]$ is the mass fraction of reactant. The reaction constants, both positive, are q—the heat release parameter and k—the reaction rate. Here, $q > 0$

corresponds to an exothermic reaction. The diffusion coefficients B and D are also assumed to be positive constants. Finally, we define ϕ by

$$\phi(u) = \begin{cases} 0, & \text{if } u \leq u_{\text{ig}}\,, \\ \mathrm{e}^{-\mathcal{E}_A/(u-u_{\text{ig}})}, & \text{if } u > u_{\text{ig}}\,, \end{cases} \tag{3}$$

where the positive parameter $\mathcal{E}_A$ is the activation energy, and u_{ig} is a fixed lower ignition threshold.Our interest is in traveling-wave solutions; these are solutions of the form

$$u(x,t) = \hat{u}(\xi), \quad z(x,t) = \hat{z}(\xi)\,, \quad \xi = x - st\,, \tag{4}$$

which connect an unburned state $(u_+, z_+) = (u_+, 1)$ as $\xi \to +\infty$ to a completely burned state $(u_-, z_-) = (u_-, 0)$ as $\xi \to -\infty$. These waves are classified as one of three types; see Table 1. Notably, from the

Table 1: Classification of detonation waves.

Strong	$u_- > s > u_+$
Weak	$s > u_-, u_+$
Chapman–Jouguet	$u_- = s > u_+$

point of view of the scalar conservation law (the inviscid Burgers equation), the shock triple (u_+, u_-, s) for a strong detonation is of Lax type, while the weak detonation is undercompressive. This feature of the weak detonation enters in an essential way in the Green function analysis [8]. Our analysis does not treat the interesting Chapman–Jouguet case.

The existence problems for strong and weak detonations may both be visualized as a connecting orbit problem for a heteroclinic orbit in a three-dimensional phase space. However, the existence problem for the weak detonation is delicate; as a manifestation of the undercompressive nature of this wave, the desired heteroclitic orbit is a structurally unstable connection corresponding to the intersection of the one-dimensional unstable manifold exiting the burned end state with the two-dimensional stable manifold entering the unburned state in the phase space $\mathbb{R}^3$. After a scaling, we find that the traveling-wave solutions of interest are steady solutions of the nonlinear system of partial differential equations

$$u_t - u_\xi + \left(u^2/2\right)_\xi = u_{\xi\xi} + qk\phi(u)z\,, \tag{5a}$$

$$z_t - z_\xi = Dz_{\xi\xi} - k\phi(u)z\,. \tag{5b}$$

Linearizing about the steady solution $(\hat{u}, \hat{z})$, we obtain the linearized equations. These equations describe the approximate evolution of a perturbation. The Green function for this system (linear but not constant coefficient) is the main object of study in the development of Proposition 1. Although we do not obtain exact formulas for the Green function, the sharp pointwise estimates are extremely useful as a means to describe the evolution of a perturbation.

3 Eigenvalues

We turn now to the spectral stability analysis. Suppose that $(\hat{u}, \hat{z})$ is a strong detonation. Then, the associated eigenvalue equations obtained by linearizing, after some manipulations, take the following form.

$$\lambda w = (1 - \hat{u})w' + q\hat{u}z + q(D-1)z' + w''\,, \tag{6a}$$

$$\lambda z + k(\phi(\hat{u}) - q\phi'(\hat{u})\hat{z})z = z' + k\phi'(\hat{u})\hat{z}w' + Dz''\,. \tag{6b}$$

The next proposition, obtained from (5) by means of an energy estimate, shows how the size of possible unstable eigenvalues is related to the shape of the profile $(\hat{u}, \hat{z})$.

Proposition 2 (High-frequency bounds) *Any eigenvalue λ of (5) with nonnegative real part satisfies*

$$\operatorname{Re}\lambda + |\operatorname{Im}\lambda| \leq \max\left\{4, \frac{1}{4D} + \left(\frac{1}{4} + \frac{1}{2}|D-1|^2\right)kL + kM\right\} \tag{7}$$

where

$$L := \sup_{x\in\mathbb{R}} \phi'(\hat{u}(x))\hat{z} \quad \textit{and} \quad M := \sup_{x\in\mathbb{R}} \big((1+q)\phi'(\hat{u})\hat{z} - \phi(\hat{u})\big)\,. \tag{8}$$

Remark 2 *A similar result holds for weak detonations. In fact, our result for weak detonations gives a tighter bound on possible unstable eigenvalues do to the monotonicity of the u-component of the viscous profile. See [3] for details.*

4 Evans-function computation and spectral stability

Finally, we outline the steps involved in the numerical computation of E.

Step 1. Profile The traveling-wave equation is a nonlinear two-point boundary-value problem posed on the whole line. To compute an approximation of the profile, it is necessary to truncate the problem to a finite computational domain $[-X_-, X_+]$. We use MATLAB's boundary-value solver, an adaptive Lobatto quadrature scheme, and we supply appropriate projective boundary conditions at $X_\pm$. The values for plus and minus spatial computational infinity, $X_\pm$, must be chosen with some care. Writing the traveling-wave equation as $\hat{U}' = F(\hat{U})$ together with the condition that $\hat{U} \to U_\pm$ as $\xi \to \pm\infty$, the typical requirement is that $X_\pm$ should be chosen so that $|\hat{U}(\pm X_\pm) - U_\pm|$ is within a prescribed tolerance of 10^{-3}.

We remark also that most of the profile solutions were found by continuation as it would have been difficult otherwise to provide an easy starting guesses to the boundary-value solver. Thus, an important aspect of the computational Evans-function approach we use here is the ability to continue the profile solutions throughout parameter space.

The connection problem for weak detonations requires additional effort. As noted above, a weak detonation profile is structurally unstable, so the problem's natural formulation does not have the correct number of boundary conditions. To overcome this, we inflate the state space, and the solver selects parameter values for which profiles exist; see [3] for details.

Step 2. High-frequency bounds Upon the completion of the first step, the next task is to compute the high-frequency spectral bounds supplied by Proposition 2. This amounts to the evaluation of the quantities M and L in (7). With these quantities in hand, we may choose a positive real number R sufficiently large that there are no eigenvalues outside of the domain

$$B_R^+ := B(0,R) \cap \{\operatorname{Re}\lambda \geq 0\}\,.$$

We have thus reduced the problem of verifying the Evans condition $(\star)$ to the problem of showing that the Evans function does not vanish[1] in the bounded region B_R^+.

[1] The zero at the origin has been removed.

Step 3. Evans function The evaluation of the Evans function is accomplished by means of the STABLAB package, a MATLAB-based package developed for this purpose. The computation of E is not immediately straightforward. One needs to compute analytically varying subspaces of growing and decaying solutions of a linear (but not constant coefficient) system of differential equations that depend on the spectral parameter. Substantial work has been done in recent years to address the computational challenges; we mention only the polar-coordinate ("analytic orthogonalization") method because it was used for our Majda model calculations. See [5] and the references therein.

Step 4. Winding Finally, we compute the number of zeros of the Evans function E inside the semicircle $S = \partial B_R^+$ by computing the winding number of the image of the curve S, traversed counterclockwise, under the analytic map E. To do this, we simply choose a collection of test λ-values on the curve S, and we sum the changes in $\arg E(\lambda)$ as we travel around the semicircle. These changes can be computed directly via the simple relation

$$\operatorname{Im} \log E(\lambda) = \arg E(\lambda) \mod 2\pi \,.$$

We test a posteriori that the change in the argument of E is less than 0.2 in each step, and we add test values if necessary to achieve this. Most curves resolved well within that tolerance using 120 mesh points in the first quadrant and reflecting by conjugate symmetry for the fourth quadrant.

5 Discussion

The stability of weak and strong viscous detonation-wave solutions of the Majda model can be treated in a common framework. Indeed, a virtue of this methodology is that it can be extended, at least in part, to the relevant physical system [1]. However, since all of the waves tested are nonlinearly stable, and the physical system is known to have quite delicate instability characteristics, it makes sense to imagine ways to modify the model so that it might more accurately reflect the behaviors known to occur in the physical system.

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