Nonlinear Two-time-scale Perturbation Theory for Transverse Combustion Dynamics

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1 Summary

Nonlinear, transverse-mode, combustion instability is examined via a two-time-variable amplitude-perturbation expansion [1]. Following an established process, a two-dimensional, unsteady chamber-wave-dynamics model [2] is used where the three-dimensional equations are integrated over the axial direction. Nonlinear, transverse-wave oscillations in the circular combustion chamber are examined with the primary flow in the axial direction. The analysis is first generalized to match a variety of relevant injection and combustion mechanisms. Then, a specific example with liquid-propellant-rocket-motor co-axial injectors is used to demonstrate the matching process between wave dynamics and the injection and combustion mechanisms. Turbulent mixing of gaseous propellants with co-axial injection and a multi-orifice, short thrust nozzle are considered, producing a characteristic time for mixing and a time lag in the energy release rate relative to pressure. The coupled combustion process and wave dynamics are calculated for a multi-injector chamber. In particular, the first-tangential mode is examined. Two coupled first-order ordinary differential equations (ODEs) are developed and solved to predict amplitude and phase-angle variations in the slow time for the major eigenfunction component of the waveform. Limit cycles and transient behaviors are resolved. Nonlinear triggering can occur in certain operational domains; above a critical initial amplitude, the amplitude grows; otherwise, it decays with time. The reduction to ODEs provides a foundation for future work on active controls.

2 Analytical Approach

It is useful to cast the two-dimensional wave dynamics equation in cylindrical polar coordinates because of the combustion chamber shape. r and θ represent radial distance from the chamber centerline and azimuthal position, respectively. The velocity components are u_r and u_θ. [2] Two time scales will be introduced [1]; a fast time scale $z = \omega t$ on which the oscillations occur and a slow time scale $\tau = \sigma t$ on which amplitudes and phase slowly change. ω is the angular frequency of the oscillation and σ is a small positive quantity that goes to zero as the oscillation amplitude goes to zero. Dependent variables become functions of both variables: e.g., $p(z, \tau, r, \theta)$. Then, $\partial p/\partial t = \omega \partial p/\partial z + \sigma \partial p/\partial \tau$. The non-dimensional wave dynamics equation becomes

$$\omega^2 \frac{\partial^2 p}{\partial z^2} - \left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right] = -\omega\sigma \frac{\partial^2 p}{\partial z \partial \tau} - \sigma^2 \frac{\partial^2 p}{\partial \tau^2} + (\gamma - 1)\left[\omega \frac{\partial E}{\partial z} + \sigma \frac{\partial E}{\partial \tau}\right] - B p^{\frac{\gamma-1}{2\gamma}}\left[\omega \frac{\partial p}{\partial z} + \sigma \frac{\partial p}{\partial \tau}\right] + N \tag{1}$$

where B is a non-dimensional positive constant generated by the nozzle boundary condition. It is $O(M)$ with the steady-state chamber mean Mach number $M << 1$. The nonlinear acoustic terms are given as

$$N \equiv \left(p^{\frac{\gamma-1}{\gamma}} - 1\right)\left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right] + \frac{(\gamma-1)}{\gamma}\frac{1}{p}\left(\omega\frac{\partial p}{\partial z} + \sigma \frac{\partial p}{\partial \tau}\right)^2 + \gamma p^{\frac{\gamma-1}{\gamma}}\left[\frac{\partial^2 (p^{\frac{1}{\gamma}} u_r^2)}{\partial r^2} + \frac{2}{r}\frac{\partial (p^{\frac{1}{\gamma}} u_r^2)}{\partial r} + \frac{2}{r}\frac{\partial^2 (p^{\frac{1}{\gamma}} u_r u_\theta)}{\partial r \partial \theta} + \frac{2}{r^2}\frac{\partial (p^{\frac{1}{\gamma}} u_r u_\theta)}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 (p^{\frac{1}{\gamma}} u_\theta^2)}{\partial \theta^2} - \frac{1}{r}\frac{\partial (p^{\frac{1}{\gamma}} u_\theta^2)}{\partial r}\right] \tag{2}$$

The quantity $E(z, \tau, r, \theta)$ represents the non-dimensional energy release rate per unit volume due to combustion and can be expected to go to zero as $M \to 0$. Furthermore, $\partial E/\partial t \to 0$ as $\varepsilon \to 0$ and / or as $M \to 0$.

The non-dimensional momentum equations become

$$\omega \frac{\partial u_r}{\partial z} + \frac{1}{\gamma}\frac{\partial p}{\partial r} = -\sigma \frac{\partial u_r}{\partial \tau} - \left[u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r}\right] + \frac{1}{\gamma}\left[1 - \frac{1}{p^{\frac{1}{\gamma}}}\right]\frac{\partial p}{\partial r} \tag{3}$$

and

$$\omega \frac{\partial u_\theta}{\partial z} + \frac{1}{\gamma r}\frac{\partial p}{\partial \theta} = -\sigma \frac{\partial u_\theta}{\partial \tau} - \left[u_r \frac{\partial u_\theta}{\partial r} + u_\theta \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r}\right] + \frac{1}{\gamma r}\left[1 - \frac{1}{p^{\frac{1}{\gamma}}}\right]\frac{\partial p}{\partial \theta} \tag{4}$$

The non-dimensional wall boundary conditions are

$$u_r(z, \tau, 1, \theta) = 0 \quad ; \quad \frac{\partial p}{\partial r}(z, \tau, 1, \theta) = p^{\frac{1}{\gamma}} u_\theta^2 \tag{5}$$

An implicit constraint is the solution must remain finite without any singularity.

A perturbation series expansion is assumed. ε will be the perturbation parameter which is a measure of oscillation amplitude. We take

$$\begin{aligned}
p &= 1 + \varepsilon p_1(z, \tau, r, \theta) + \varepsilon^2 p_2(z, \tau, r, \theta) + \varepsilon^3 p_3(z, \tau, r, \theta) + O(\varepsilon^4) \\
u_r &= \varepsilon u_{r,1}(z, \tau, r, \theta) + \varepsilon^2 u_{r,2}(z, \tau, r, \theta) + \varepsilon^3 u_{r,3}(z, \tau, r, \theta) + O(\varepsilon^4) \\
u_\theta &= \varepsilon u_{\theta,1}(z, \tau, r, \theta) + \varepsilon^2 u_{\theta,2}(z, \tau, r, \theta) + \varepsilon^3 u_{\theta,3}(z, \tau, r, \theta) + O(\varepsilon^4) \\
E &= 1 + \varepsilon E_1(z, \tau, r, \theta) + O(\varepsilon^2) \quad ; \quad \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3) \\
p^{\Gamma(\gamma)} &= 1 + \varepsilon \Gamma p_1 + \varepsilon^2[\Gamma p_2 + \tfrac{\Gamma(\Gamma-1)}{2} p_1^2] + O(\varepsilon^3)
\end{aligned} \tag{6}$$

The zeroeth-order solutions are the steady-state solutions; thus, the non-dimensional $p_0 = 1$ and $u_{r,0} = u_{\theta,0} = 0$. It will be shown that $\omega_1 = 0, \sigma = M = \varepsilon^2$. For simplicity, those values will be taken now and proven later.

Now, we substitute the series into Equations (1) through (5) and separate according to powers of ε.

An infinite number of modes are possible. We select here the very common first-tangential spinning (i.e., travelling in the azimuthal direction). The first-order wave equation for p_1 is the classical homogeneous wave equation, yielding the well known results:

$$p_1 = A(\tau)J_1(s_{11}r)cos(z-\theta+\psi(\tau)) \ ; \ u_{r1} = -\frac{A}{\gamma s_{11}}\frac{dJ_1}{dr}sin(z-\theta+\psi) \ ;$$
$$u_{\theta 1} = \frac{A}{\gamma s_{11}r}J_1 cos(z-\theta+\psi) \ ; \ \omega_0 = s_{11} = 1.8413 \quad (7)$$

J_n, A, and ψ are the Bessel function of first kind and n^{th} order, slowly varying amplitude, and slowly varying phase angle.

The second-order quantity p_2 is governed by a linear non-homogeneous wave equation where the forcing function is known based on the first-order solution. The forcing function is

$$= -A^2\frac{Q_0(r)}{r^2} - A^2\frac{Q_2(r)}{r^2}cos(2[z-\theta+\psi]) - A^2\frac{Q_{2s}(r)}{r^2}sin(2[z-\theta+\psi])$$

where Q_0, Q_2, and Q_{2s} are functions of squares and products of Bessel functions. To prevent a resonant singularity, part of the forcing function has been forced to zero value by setting $\omega_1 = 0$. For this solution, we now have the sum of three particular solutions:

$$p_2 = A^2F_0(r) + A^2F_2(r)cos(2[z-\theta+\psi]) + A^2F_{2s}(r)sin(2[z-\theta+\psi]) \quad (8)$$

where the first particular solution $F_0(r)$ is

$$F_0 = K + \frac{J_1^2}{2\gamma} + \frac{1}{2\gamma}\int_0^r \left[\frac{J_2^2}{r'} - \frac{2J_1J_2}{s_{11}(r')^2}\right]dr' \quad (9)$$

The constant K is determined because the volume integral of density yields the same mass as given in steady-state operation. Thus, we have

$$K = \frac{\gamma-3}{2\gamma}\int_0^1 rJ_1^2 dr + \frac{1}{\gamma}\int_0^1\left[\int_0^r\left[\frac{2J_1J_2}{s_{11}(r')^2} - \frac{J_2^2}{r'}\right]dr'\right]rdr \quad (10)$$

The second particular solution F_2 is given as

$$\begin{aligned} F_2(r) &= Y_2(2s_{11}r)\int_0^{2s_{11}r}\frac{J_2(z)Q_2(z/(2s_{11}))}{W(z)[z/(2s_{11})]^2}dz - J_2(2s_{11}r)\int_0^{2s_{11}r}\frac{Y_2(z)Q_2(z/(2s_{11}))}{W(z)[z/(2s_{11})]^2}dz \\ &= 2\pi s_{11}^2\left[Y_2(2s_{11}r)\int_0^r\frac{J_2(2s_{11}r')Q_2(r')}{r'}dr' - J_2(2s_{11}r)\int_0^r\frac{Y_2(2s_{11}r')Q_2(r')}{r'}dr'\right] \end{aligned} \quad (11)$$

The first term has a singularity at $r = 0$ introduced through Y_2, the Bessel function of the second kind. However, the product with multiplying integral will remove the singularity. In the second term, the product also removes a singular behavior.

The remaining second-order particular solution (F_{2s}) is

$$F_{2s}(r) = 2\pi s_{11}^2\left[Y_2(2s_{11}r)\int_0^r\frac{J_2(2s_{11}r')Q_{2s}(r')}{r'}dr' - J_2(2s_{11}r)\int_0^r\frac{Y_2(2s_{11}r')Q_{2s}(r')}{r'}dr'\right] \quad (12)$$

Solution for $u_{\theta 2}$ requires integration over z. The "constant" of integration (actually allowed to be a function of r) is set to zero by the condition of zero vorticity (zero circulation) to this order of the perturbation series. The solutions for u_{r2} and $u_{\theta 2}$ become

$$\begin{aligned} u_{r2} &= A^2G_2(r)sin(2[z-\theta+\psi]) + A^2G_{2s}(r)cos(2[z-\theta+\psi]) \\ G_2(r) &\equiv \frac{1}{2\gamma^2s_{11}^2}\left[\frac{J_1J_2}{r^2} - \frac{s_{11}J_2^2}{2r} - \gamma s_{11}\frac{dF_2}{dr}\right] \ ; \ G_{2s}(r) \equiv \frac{1}{2\gamma s_{11}}\frac{dF_{2s}}{dr} \end{aligned} \quad (13)$$

$$\begin{aligned} u_{\theta 2} &= A^2H_2(r)cos(2[z-\theta+\psi]) + A^2H_{2s}(r)sin(2[z-\theta+\psi]) \\ H_2(r) &\equiv \frac{1}{\gamma s_{11}r}\left[F_2 + \frac{J_1^2-J_2^2}{4\gamma} + \frac{J_1J_2}{2\gamma s_{11}r}\right] \ ; \ H_{2s}(r) \equiv \frac{1}{\gamma s_{11}r}F_{2s} \end{aligned} \quad (14)$$

3 Solutions for Amplitude and Phase

We obtain the two following differential equations by setting two portions of the third-order forcing function to be zero in order to avoid resonant singularities:

$$\frac{dA}{d\tau} = \quad k_1 A + k_2 A^3 \quad ; \quad \frac{d\psi}{d\tau} = -2\omega_2 - k_3 - k_4 A^2 \tag{15}$$

$$k_1 \equiv (\gamma - 1)\bar{E}_{c,1} - B = (\gamma - 1)\frac{\int_0^1 E_{c,1}(r) J_1(s_{11} r) r dr}{\int_0^1 J_1^2(s_{11} r) r dr} - B$$

$$k_2 \equiv B\left[\frac{\gamma - 1}{4\gamma}\frac{\int_0^1 (2F_0 J_1 + F_2 J_1) r dr}{\int_0^1 J_1^2 r dr} + \frac{\gamma^2 - 1}{32\gamma^2}\frac{\int_0^1 J_1^4 r dr}{\int_0^1 J_1^2 r dr}\right] - \frac{\int_0^1 q_{1s}(r) J_1(s_{11} r) r dr}{s_{11}\int_0^1 J_1^2(s_{11} r) r dr} \quad ;$$

$$k_3 \equiv (\gamma - 1)\bar{E}_{s,1} = (\gamma - 1)\frac{\int_0^1 E_{s,1}(r) J_1(s_{11} r) r dr}{\int_0^1 J_1^2(s_{11} r) r dr} \quad ; \quad k_4 \equiv \frac{\int_0^1 q_1(r) J_1(s_{11} r) r dr}{s_{11}\int_0^1 J_1^2(s_{11} r) r dr}$$

The sum of $E_{c,1}(r)$ multiplied a cosine function of $z - \theta + \psi$ plus $E_{s,1}(r)$ multiplied a sine function of $z - \theta + \psi$ gives the first perturbation of the energy release rate per unit volume. $q_1(r)$ and $q_{1s}(r)$ are determined from the forcing function of the third-order wave equation and depend on triple products of known Bessel functions. Analytical solutions can be found for these two first-order ordinary differential equations. With no loss of generality, $A = A_0$ and $\psi = 0$ for the initial values. For the first equation, the integrated solution becomes

$$\frac{A(\tau)}{A_0} = \left[(1 + \frac{k_2}{k_1}A_0^2)e^{-2k_1\tau} - \frac{k_2}{k_1}A_0^2\right]^{-1/2} \tag{16}$$

Consider first the cases where k_1 and k_2 have identical signs. If $k_1 < 0$ and $k_2 < 0$ (Case I), the solution for A goes to zero value as $\tau \to \infty$. If instead $k_1 > 0$ and $k_2 > 0$ (Case II), the solution for A goes to infinity in a finite time. Under this condition of unconditional instability in Case II with both k_1 and k_2 having positive values, a stable limit cycle is expected in practice. However, the perturbation series has not yet captured sufficiently high powers of ε to predict the stable limit cycle. So, the solution is artificially predicted to grow to infinite amplitude in a finite time; rather, it is expected to grow to a finite stable amplitude in an infinite time if higher order analysis were applied.

If k_1 and k_2 have opposite signs, a limit cycle clearly exists at $A = A* \equiv \sqrt{-k_1/k_2}$ with zero-valued time derivative. For a more informative display, we may rewrite Equation (16) as

$$\frac{A(\tau)}{A_0} = \left[(1 - (\frac{A_0}{A*})^2)e^{-2k_1\tau} + (\frac{A_0}{A*})^2\right]^{-1/2} \tag{17}$$

If $k_1 < 0, k_2 > 0$, and $A_0 < A*$ (Case IIIa), the solution for A decays to zero value as $\tau \to \infty$; while the solution for A grows to infinity in a finite time if $k_1 < 0, k_2 > 0$, and $A_0 > A*$ (Case IIIb). (Note that mathematically in either Case IIIa or IIIb, the value of $A*$ is approached as $\tau \to -\infty$.) In Case III here, the limit cycle at $A*$ is unstable; a stable limit cycle should exist at a higher value of A but the truncated perturbation series does not reveal it. So, again the predicted growth to infinity in a finite time is artificial; rather, growth in an infinite time to a finite stable value is expected.

If $k_1 > 0, k_2 < 0$, and $A_0 < A*$ (Case IVa), the solution grows with $A \to A*$ as $\tau \to \infty$ and $A \to 0$ as $\tau \to -\infty$. If $k_1 > 0, k_2 < 0$, and $A_0 > A*$ (Case IVb), the solution decays with $A \to A*$ as $\tau \to \infty$. In Case IV here, the limit cycle at $A*$ is stable.

k_1 and k_2 must have opposite signs to predict a limit cycle. If k_1 were positive (negative) with a negative (positive) k_2, a stable (unstable) limit cycle is predicted. The positive k_1 with negative k_2 implies unconditional instability, any small perturbation grows to some stable limit cycle. The reversed signs indicate a bi-stable behavior with conditional stability; however, the expected stable limit cycle at an amplitude greater than the unstable limit-cycle amplitude $A*$ is not predicted. Presumably, the perturbation analysis must be carried to higher order for that prediction.

If k_1 and k_2 are both positive, any disturbance grows indicating unconditional instability without prediction of a required stable limit-cycle. Again, we presume higher-order analysis is required here. If k_1 and k_2 were both negative, any disturbance decays indicating unconditional stability within the limits of our analysis; although a higher-order solution could indicate a bi-stable character. For example, the addition of an $O(A^5)$ term with a positive coefficient to the right-side of Equation (15) for A would lead to prediction of an unstable limit cycle.

ω_2 is the frequency perturbation that applies in the limit cycle where ψ ceases to vary with time; thus, its value can be determined to be $\omega_2 = -(k_3 + k_4 A*^2)/2$. Note that ω_2 can be simply ignored where a limit cycle is not found.

The second equation in (15) is readily solved by integration of a simple quadrature after substitution for A using Equation (16).

$$\psi = k_4 A*^2\,\tau + \frac{k_4}{2k_2} ln\left[(\frac{A_0}{A})^2 e^{2k_1\tau}\right] = k_4 A*^2\,\tau - k_4 A*^2\,\tau + \frac{k_4}{2k_2} ln\left[(\frac{A_0}{A})^2\right] = \frac{k_4}{2k_2} ln\left[(\frac{A_0}{A})^2\right]$$

$$A \to A* \Rightarrow \psi \to \frac{k_4}{2k_2} ln\left[(\frac{A_0}{A*})^2\right] \tag{18}$$

These results provide useful information for the nonlinear regimes where amplitude is not too large. As noted, higher order expansion terms might be needed in some cases; however, the pattern has been identified of a polynomial with differences in consecutive terms being $O(A^2)$.

4 Analysis of Mixing and Combustion with N Co-axial Injectors

We consider the specific example of N co-axial injectors (oxygen surrounded by fuel) with individual flames at each injector. Only the linearized perturbations will appear in matching the wave dynamics to third order. The Oseen approximation and use of Green's functions allow an analytical solution. The definitions are given that

$$\begin{aligned} V_3 &\equiv \frac{\gamma-1}{\gamma}\left(\int_0^{L_f} \frac{d\bar{E}}{dx}\left[1 - \frac{T_{ss,\infty}}{\bar{T}_f(x)}\right]dx + \left[1 - \frac{T_{i0}}{T_{ss,\infty}}\right]\int_0^{L_f} \frac{T_{ss,\infty}}{\bar{T}_f(x)} cos(s_{11}x/\bar{U})\frac{d\bar{E}}{dx}V_2(x)dx\right) \\ V_4 &\equiv \frac{\gamma-1}{\gamma}\left[1 - \frac{T_{i0}}{T_{ss,\infty}}\right]\int_0^{L_f} \frac{T_{ss,\infty}}{\bar{T}_f(x)} sin(s_{11}x/\bar{U})\frac{d\bar{E}}{dx}V_2(x)dx \end{aligned} \tag{19}$$

The two integrals with the sinusoidal oscillations of a kinematic wave with short wavelength will have lower values than the first integral. The steady-state flame temperature T_f and the function V_2 are determined analytically.

We must redistribute the burning rates for all N injectors in an eigenfunction series. We are most interested in the Fourier sine and cosine functions with the fundamental frequency. Some convenient definitions are

$$
\begin{aligned}
a_i &\equiv \frac{\int_0^{2\pi}\int_0^1 \delta(r-r_i,\theta-\theta_i)J_1(s_{11}r_i)cos\theta_i J_1(s_{11}r)cos\theta\; rdrd\theta}{\pi\int_0^1 J_1^2(s_{11}r)\; rdr} = \frac{J_1^2(s_{11}r_i)cos^2\theta_i}{\pi\int_0^1 J_1^2(s_{11}r)\; rdr} \\
b_i &\equiv \frac{\int_0^{2\pi}\int_0^1 \delta(r-r_i,\theta-\theta_i)J_1(s_{11}r_i)sin\theta_i J_1(s_{11}r)sin\theta\; rdrd\theta}{\pi\int_0^1 J_1^2(s_{11}r)\; rdr} = \frac{J_1^2(s_{11}r_i)sin^2\theta_i}{\pi\int_0^1 J_1^2(s_{11}r)\; rdr} \\
A_i &\equiv a_i + b_i = \frac{J_1^2(s_{11}r_i)}{\pi\int_0^1 J_1^2(s_{11}r)\; rdr}
\end{aligned}
\tag{20}
$$

The key Fourier coefficients in the expansion, account for contributions from all N injectors, are

$$
\bar{E}_{c,1} = \frac{V_3}{2}\left(\sum_{i=1}^{N} A_i\right) \quad ; \quad \bar{E}_{s,1} = \frac{1}{M}\frac{V_4}{2}\left(\sum_{i=1}^{N} A_i\right) \tag{21}
$$

After integration to produce components in a proper Fourier-Bessel series, we have a useful measure of the fluctuation in the time rate of energy per unit volume.

These results may now be used for substitution in Equations (15) to evaluate the important constants.

5 Concluding Remarks

The theory predicts that the limit-cycle mean-to-peak dimensional pressure amplitude scales roughly as $p_{steady-state}/\sqrt{M}$ assuming a weak dependence of the combustion on the mean-flow Mach number and a small influence of Mach number dependence appearing through the B term. The Mach number dependence of the amplitude comes primarily from k_2. The Mach number at the nozzle entrance scales roughly as throat area A_t for $M << 1$ while steady-state chamber pressure scales as the reciprocal of A_t. Therefore, one can vary A_t at constant mass flow and show that dimensional pressure amplitude is proportional to $A_t^{-3/2}$. Thereby, for example, a thirty per cent change in nozzle throat area produces a nearly fifty percent change in the mean-to-peak limit-cycle amplitude. The dimensional frequency perturbation at the limit cycle, should vary with M (or A_t).

Here, the waveform consists of a basic resonant mode of oscillation with the superposition of higher harmonics on a fundamental mode. The method could be used for other tangential, radial, and mixed radial-tangential modes, including both standing and travelling modes. Situations where more than one fundamental mode appear with non-integer frequency ratios would produce ”wobbly” waveforms where the solutions cannot be expressed in terms of one frequency. Energy transfer between these fundamental resonant modes would occur and sub-harmonics might be produced.

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