Kinetics of Drop Shattering Behind Detonation Wave

A. G. Girin
Odessa National Maritime University
Odessa, Ukraine

1 Introduction

The shattering of drops is a key process in heterogeneous detonation phenomenon. It sets a sizes, quantity and moments of tearing-off of stripped daughter droplets. Just these parameters control following motion and evaporation of a lot of finest droplets, which are gone with a gas flow into wake of parent drop and produce a two-phase combustible spray there. To determine kinetic regularities of shattering experimentally is too complex problem, because the event picture is shadowed by dense mist of finest particles and their vapors. So, the regularities must be found by means of theoretical modelling of the process. To construct a theory of detonation in gas-droplets systems one must build first of all a model of shattering process which would be able to give droplets sizes and moments of their tearing-off in order to determine quantitatively further processes of stripped mass motion, evaporation and combustible mixture formation. Thus, distribution function of quantity of daughter droplets by sizes and its evolution in space and time must be the basic element of mathematical model of heterogeneous detonation wave.

An investigation of local instability of drop surface with due regard to changing of velocity profiles across conjugated boundary layers, as well as to changing of boundary layers thicknesses and velocities along drop surface [1] revealed for weak-viscosity liquids a new type of hydrodynamic instability – so-called ”gradient instability”. As distinct from Kelvine – Helmholtz type, which is grounded on pressure difference action, mechanism of gradient instability consists in action of large gradient of inertia forces, caused by huge velocity gradient ($10^5 - 10^7 sec^{-1}$), in curvilinear flow inside perturbed liquid boundary layer. The mechanism of gradient instability explains the ”stripping” mode of shattering as quasi-continuous dispersing and predicts all the main features of event in speedy flows [2]. Its application permits below to obtain law of drop mass history $m(t)$ (ablation law) and distribution function of daughter droplets, stripped to any time moment.

2 Equation for daughter droplets quantity

Dependencies of wavenumber $\Delta_{\rm m}$ and increment $\mathrm{Im}(z_{\rm m})$ of dominant unstable disturbance from ”surface” Webber number $\mathrm{We_s} = \rho_g V_{\rm s}^2 \delta_l / \sigma$ show [2], that there exists a critical point $\varphi_{\rm cr}(t)$ on drop surface, which divides it on stable $\varphi < \varphi_{\rm cr}$ and unstable $\varphi > \varphi_{\rm cr}$ parts. Assuming the potential streamlining of drop and taking gas boundary layer thickness in the form of Ranger $\delta_{\rm g}(\varphi, t) = 2.2R(t){\rm Re}^{-0.5}(t)\Psi(\varphi)$, $\Psi(\varphi) \equiv \left((6\varphi - 4\sin 2\varphi + 0.5\sin 4\varphi)/\sin^5\varphi\right)^{0.5}$, using continuity of viscous tangential tension on drop surface as conjugation condition, we obtain condition for gradient instability to exist on drop surface:

$$\frac{2.475\ \alpha}{(1+\alpha^{\xi})^2}\sqrt{\tilde{R}(\tau)\,(1-W(\tau))^3}\,\sin^2\varphi\ \Psi(\varphi)\ {\rm GI}\ >\ 0.004, \tag{1}$$

where ${\rm GI} = {\rm We}_\infty/{\rm Re}_\infty^{0.5}$ is criterion of gradient instability, $\tilde{R} = R/R_0$ and $W = w/V_\infty$ are dimensionless drop radius and velocity, $\tau = t/t_{\rm ch}$, $t_{\rm ch} = 2R_0/\sqrt{\alpha}V_\infty$, V_∞ – velocity of gas flow, $\alpha = \rho_\infty/\rho_l$, $\mu = \mu_\infty/\mu_l$, $\xi = \log_\alpha(\alpha\,\mu)^{1/3}$ is parameter of mutual viscous engagement of media in boundary layers. Equality in (1) defines the value of $\varphi_{\rm cr}$; at $\alpha\,{\rm GI} > 5.25\cdot 10^{-4}$ we have $\varphi_{\rm cr} < \pi/2$ and part of drop surface adjacent to edge is unstable, providing a possibility of dispersing. The values of $\varphi_{\rm cr}$ are small enough in detonation: $\varphi_{\rm cr} \ll \pi$, so, most part of drop surface generates a mist of droplets. Condition ${\rm GI} > \simeq 0.3$ was first obtained empirically by Rabin et al. as condition of stripping breakup mode.

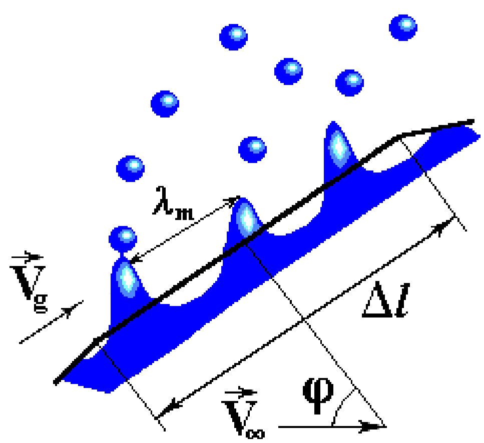

Figure 1: Scheme of dispersing on the elementary ground of drop surface

Due to axisymmetric character of streamlining of drop the quantity of unstable waves on any ground $\Delta l = R(t)\Delta\varphi$ of surface (fig. 1) equals to quantity of torus, that are torn from spherical belt defined by this ground. Let's assume that radius of torn droplet is proportional to the length of dominant unstable wave $r = k_{\rm r}\lambda_{\rm m}, k_{\rm r} < 0.25$, then, dividing volume of torus, torn in time interval $t_{\rm i} = k_{\rm t}{\rm Im}^{-1}(z_{\rm m})$, $k_{\rm t} \geq 1$, by volume of droplet, we obtain equation for droplets quantity Δn torn from the ground:

$$\Delta n(\varphi,\tau) = B_2\sqrt{\tilde{R}(\tau)(1-W(\tau))^5}\,\frac{\sin^2\varphi}{\Psi^3(\varphi)}\Delta\varphi\,\Delta\tau,\quad B_2 = \frac{0.30\Delta_{\rm m}^2({\rm We_s}){\rm Im}(z({\rm We_s})){\rm Re}_\infty^{1.5}}{\pi k_{\rm r}k_{\rm t}(1+\alpha^{\xi})}\left(\frac{\mu^2}{\alpha}\right)^{\frac{7}{6}}; \tag{2}$$

$$\tilde{r}(\varphi,\tau) = B_1 T(\tau)\Psi(\varphi),\ \ \tilde{r} = \frac{r}{R_0},\ \ T(\tau) = \left(\frac{\tilde{R}(\tau)}{(1-W(\tau))}\right)^{0.5},\ \ B_1 = \frac{3.11\pi k_{\rm r}}{\Delta_{\rm m}({\rm We_s}){\rm Re}_\infty^{0.5}}\left(\frac{\alpha}{\mu^2}\right)^{\frac{1}{3}}. \tag{3}$$

3 Equation of parent drop ablation

Dividing now the stripped mass by period of its stripping t_i, we obtain the rate of mass efflux from this ground, and by integrating along windward surface from φ_cr to $\pi/2$, with accounting of weak dependence of ratio $\mathrm{Im}(z_\mathrm{m})/\Delta_\mathrm{m}$ from $\mathrm{We_s}$, we obtain [3] the differential equation of drop ablation

$$\frac{dM}{d\tau} = -A\tilde{R}^2(\tau)\,(1 - W(\tau))\left(1 - \frac{2\varphi_{\,\mathrm{cr}}(\tau)}{\pi} + \frac{\sin 2\varphi_{\,\mathrm{cr}}(\tau)}{\pi}\right),\ \ A = \frac{0.41\pi^3 k_\mathrm{r}^2}{k_\mathrm{t}(1+\alpha^\xi)}\left(\frac{\mu^2}{\alpha}\right)^{\frac{1}{6}}, \quad (4)$$

where $M = m/m_0$. To determine $W(\tau)$ and $\varphi_\mathrm{cr}(\tau)$ it requires simultaneous integration of drop motion equation and eq. (1). But for $\varphi_\mathrm{cr} \ll \pi$ and spherical drop $M(\tau) = \tilde{R}^3(\tau)$ we obtain the ablation law

$$M(\tau) = \left(1 - A\left(\tau - \alpha^{0.5}X_\mathrm{d}(\tau)\right)/3\right)^3,\ \ \tilde{R}(\tau) = 1 - A\left(\tau - \alpha^{0.5}X_\mathrm{d}(\tau)\right)/3, \quad (5)$$

immediately, which indicates evidently the direct influence of law of drop motion $X_\mathrm{d} = X_\mathrm{d}(\tau)$ on its ablation law. Let's use now empirical data of Reinecke, Waldman [6] and write down law of drop motion in the form $\sqrt{\alpha}X_\mathrm{d}(\tau) = \tau - (1 - \exp(-H\tau))/H$, $H = 2\sqrt{\alpha}$, whence

$$W = 1 - \exp(-H\tau), \quad M = \tilde{R}^3 = (1 - h\,(1 - \exp(-H\tau)))^3, \quad (6)$$

where parameter $h = A/3H$ reflects the relation between two governing factors of shattering: the rate of mass efflux ($\sim A$) and rate of relaxational reducing of relative velocity of gas flow and parent drop ($\sim H$). As (6) shows, when $h > 1$ drop is wholly shattered to time moment $\tau_\mathrm{b} = H^{-1}\ln(h/(h-1))$; when $h < 1$ dispersion terminates before the whole drop breaks because latter factor leads to quick reducing of main reason of dispersing – relative velocity $1 - W$; remnant has radius $\tilde{R}_\mathrm{r} = 1 - h$ and it may be shattered by another mechanism, for example, by Rayleigh – Taylor instability [2]. The analysis shows, that values of h for detonative systems are slightly higher then $h = 1$, values $h \geq 4$ correspond to ablation of liquid meteoroids and case $h < 1$ – to incomplete shattering of viscous drops. Comparison of ablation law (6) with experimental data [4] indicates their good enough agreement [3].

4 Distribution function in the case when $h = 1$

To obtain distribution function $f_\mathrm{n}(\tilde{r},\tau) = \Delta n(\tilde{r},\tau)/\Delta\tilde{r}$ we need to integrate (2) along each line $\tilde{r}(\varphi,\tau) = const$ inside strip $\Delta\tilde{r} = const$; sets of these lines are different for $h > 1$ and $h < 1$ as fig. 2 shows. Dispersing process begins in base diapason $\tilde{r}_{0l} < \tilde{r} < \tilde{r}_{0r}$, which corresponds to initial interval $\varphi_\mathrm{cr}(0) < \varphi < \pi/2$ and continuous in domain A. As equation (3) shows, diapason of droplets sizes is then widened in domain B by fine fractions for $h > 1$ cases and coarse fractions for $h < 1$ cases.

In case $h = 1$ (6) gives $\tilde{R} = \exp(-H\tau) = 1 - W(\tau)$, then it follows from (3) that $T \equiv 1$, so, the lines $\tilde{r}(\varphi,\tau) = const$ are all parallel to time axis. It appears in the remarkable case $h = 1$ of equality of ablation rate to rate of reducing of relative velocity, that for each fixed ground on drop surface the reducing of stripped droplets size due to reducing in time of parent drop size is strictly compensated by its growth due to reducing of relative velocity. Thus, size of droplets that torn from fixed ground $\Delta\varphi$ remains unchanged and it can be determined at $\tau = 0$. There is nothing to do but to sum the quantity Δn in time. Eliminating $\Psi(\varphi)$ in (2) with a help of (3) and substituting $\Delta\varphi$ by $\Delta\varphi = \Delta\tilde{r}/B_1\Psi'$, we obtain:

$$\frac{\Delta n(\tilde{r},\tau)}{\Delta\tilde{r}} = f_\mathrm{n}(\tilde{r},\tau) = \frac{1 - \exp(-3H\tau)}{A\,\tilde{r}^2}\,\frac{B_1^3 B_2 \sin^3\varphi_0(\tilde{r})}{(8B_1^2 - 2.5\,\tilde{r}^2\cos\varphi_0(\tilde{r}))},\ \ \tilde{r}_{0l} < \tilde{r} < \tilde{r}_{0r}. \quad (7)$$

Formula (7) permits to calculate intermediate distribution of daughter droplets torn to any time moment $\tau < \tau_{\rm b}$. Calculated values of $\Delta n = f_{\rm n}\Delta\tilde{r}$ at $h = 1.00$ are given on fig. 3. The modal radius of

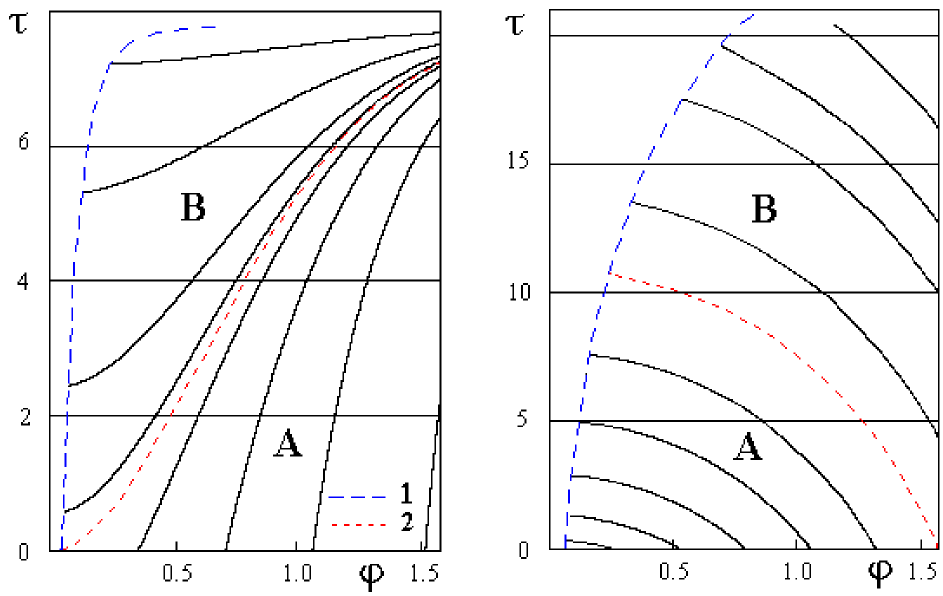

Figure 2: Set of lines $\tilde{r}(\varphi,\tau) = const$ (black, solid). Left: $h = 1.5$. Right: $h = 0.5$. Line *1* – left bound $\varphi_l = \varphi_{\rm cr}(\tau)$ of domain of dispersing; line *2* – border of domains **A** and **B**: $\tilde{r} = \tilde{r}_l$ on the left and $\tilde{r} = \tilde{r}_r$ on the right

distribution $\tilde{r}_{\rm mod}$ can be easily obtained from (7); it gives $\tilde{r}_{\rm mod} \approx 2B_1$. As well, the approximated formula for total quantity of stripped droplets $N = \sum_{\Delta r} \Delta n$ may be found in the form:

$$N \approx 0.047\, B_2 A^{-1}\,(1.45 - 0.76\,\varphi_{l0} + 0.25\sin(3.05\varphi_{l0}))\,. \tag{8}$$

5 Distribution function in general case

After eliminating $\Psi(\varphi)$, substituting $\Delta\tau = \Delta\tilde{r}/(B_1\dot{T}(\tau)\Psi(\varphi))$ and integrating (2) along $\tilde{r}(\varphi,\tau) = const$ from $\varphi_* = \varphi_0 = \varphi(0)$ to $\varphi^* = \pi/2$ (see fig. 2), we obtain the equation for distribution function

$$\Delta n = f_{\rm n}\Delta\tilde{r} = 2\frac{B_1^3 B_2}{(h-1)H\tilde{r}^4}\int\limits_{\varphi_0}^{\pi/2} \tilde{R}^3(\tau(\varphi))\,(1 - W(\tau(\varphi)))\sin^2\varphi\, d\varphi\;\Delta\tilde{r}, \tag{9}$$

where $\tau(\varphi)$ is determined for each fixed $\tilde{r}$ from (3). Equation (9) points out the influence of laws of ablation and motion of drop on distribution function. To evaluate the curvilinear integral is a knotty problem in view of $\tau(\varphi)$ kind, so, one possible way is to approximate path of integration by straight line $\tau - \tau_* = (\varphi - \varphi_*)/a_{\rm ef}$ with some effective slope $a_{\rm ef}(\tilde{r}; h)$. So we obtain from (9) using (6)

$$\Delta n(\tilde{r}) = f_{\rm n}(\tilde{r})\Delta\tilde{r} = \frac{B_1^3 B_2}{(h-1)\tilde{r}^4}\,\frac{a_{\rm ef}(\tilde{r})}{H^2}\sum_{i=1}^{4} A_i\left[\Phi_{i\,*}(\tilde{r}) - \Phi_i^*(\tilde{r})\right]\Delta\tilde{r}, \tag{10}$$

where $\Phi_i(\tilde{r}) = C^i(\tilde{r})\left(\sin^2\varphi(\tilde{r}) + \sin^2(\varphi(\tilde{r}) + \theta_i(\tilde{r}))\right)$ and $C(\tilde{r}) = (h-1)/\left(h - (\tilde{r}/(B_1\Psi(\varphi)))^2\right)$ must be calculated on lower $\varphi = \varphi_*(\tilde{r})$ and upper $\varphi = \varphi^*(\tilde{r})$ limits of integration; $A_i = 0.25C_4^i h^{i-1}(1-h)^{4-i}$, $\theta_i = \pi - \gamma_i$ at $h < 1$ and $\theta_i = \gamma_i$ at $h > 1$, $\gamma_i = \arcsin((iH/2a_{\text{ef}})^2 + 1)^{-0.5}$. Values of $\varphi_*(\tilde{r})$ and $\varphi^*(\tilde{r})$ are to be found from equations of boundaries of dispersion region $\varphi_l = \varphi_{\text{cr}}(\tau)$, $\varphi_r = \pi/2$ (see fig. 2) and of lines $\tilde{r}(\varphi, \tau) = const$. Analysis of behavior of these lines permitted to find expression for a_{ef}, which is valid in wide diapason of h. As quantity of stripped droplets decreases with time (see (2)), effective slope must be fitted with account of most influence of its initial values and less influence of its mean values $a_{\text{mv}} = (\varphi^* - \varphi_*)/(\tau^* - \tau_*)$. Besides that, it is necessary to set natural demand to get in the limit $h \to 1$ the exact expression (7), obtained in case $h = 1$. Eventually we came to following expressions for a_{ef} in domains A and B:

$$a_{\text{ef A}} = \frac{\left(h - 1 + h^{-2} + k(h)\,|h-1|^{0.5}\,h^{-1}\right)a_{\text{mv}}a_*}{(h-1)a_* + a_{\text{mv}}h^{-2} + (a_{\text{mv}} + a_*)\,|h-1|^{0.5}\,h^{-1}}, \quad a_{\text{ef B}} = \frac{\left(k_1 h + k_2 h^{-3}\right)a_{\text{mv}}a_*}{ha_* + a_{\text{mv}}h^{-3}} \quad (11)$$

with $k(h) = 0.93(2h-1)h^{-1}$, $k_1 = 1.13$, $k_2 = 0.87$ for $h > 1$ and $k(h) = 1.33h^{-0.5}$, $k_1 = 0.80$, $k_2 = 1.13$ for $h < 1$. The calculated by formulae (7), (10)–(11) distributions $\Delta n(\tilde{r})$ are shown on fig. 3.

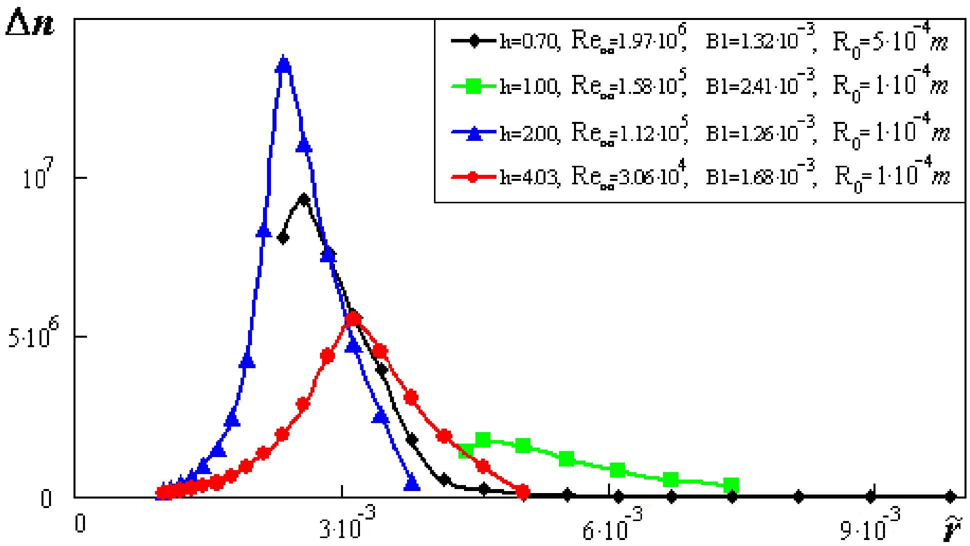

Figure 3: Distributions $\Delta n(\tilde{r})$ at various h, Re_∞

6 Peculiarities of droplets distributions

The most parts of mass and quantity of daughter droplets at any h are generated in a base diapason $\tilde{r}_{0l} < \tilde{r} < \tilde{r}_{0r}$ (domain A). At $h > 1$ function $\Delta n(\tilde{r})$ has ascending and descending branches, which make maximum at $\tilde{r}_{\text{mod}}$. It appears due to small rate of production of droplets to the left of $\tilde{r}_{\text{mod}}$, while to the right – small is the period of existence of conditions for such a production (fig. 2). The shape of curve $\Delta n(\tilde{r})$ depends on values of h, as illustrated by fig. 3, while bench-mark values of $\tilde{r}_{0l}$, $\tilde{r}_{0r}$, $\tilde{r}_{\text{mod}}$, $\Delta n(\tilde{r}_{\text{mod}})$ depend on B_1, B_2. In accordance with (2), (3) the sizes of totality of daughter droplets are defined by parameter $B_1 \sim \mathrm{Re}_\infty^{-1/2}\alpha^{1/3}\mu^{-2/3}$, which plays the role of scale of sizes; as well, $B_2 \sim \mathrm{Re}_\infty^{3/2}\alpha^{-7/6}\mu^{7/3}$ is responsible for quantity scale. As $h \sim \alpha^{-2/3}\mu^{1/3}$ growth, the part of fine fractions widens and at $h \simeq 2$ it becomes comparable with that of the coarse one (fig. 3).

7 Distribution function for theoretical law of drop motion

Distribution function (10) was derived on a base of empirical law of drop motion (6), well enough fitted quantitatively. To eliminate arbitrary influence from the law's form, the procedure of obtaining $f_\mathrm{n}(\tilde{r})$ was undertaken in the similar way, but it was grounded now on simultaneous integrating of differential equation of drop motion and equation of ablation (4), which for natural $\eta = 3h/(h-1)$ gives

$$\tilde{R}(\tau) = D^h, \quad W(\tau) = 1 - D, \quad f_\mathrm{n}(\tilde{r}) = \frac{3hB_1^3 B_2}{(h-1)A\tilde{r}^4}\left[\frac{1}{c(\eta+1)}P_{\eta+1}(\varphi) - F_\eta(\varphi)\right]_{\varphi_*}^{\varphi^*} \qquad (12)$$

where $F_\eta(\varphi) = \frac{\sin 2\varphi}{2}\sum_{k=0}^{\mathrm{E}(\eta/2)}(-0.25)^k P_\eta^{(2k)}(\varphi) + \frac{\cos 2\varphi}{4}\sum_{k=1}^{\mathrm{E}((\eta+1)/2)}(-0.25)^{k-1} P_\eta^{(2k-1)}(\varphi)$, $P_\eta(\varphi) = (b + c\varphi)^\eta$, $P_\eta^{(k)}$ – its k-th derivative, $b = 1 - (h-1)C(\tau_* - \varphi_*/a_\mathrm{ef})$, $c = -(h-1)C/a_\mathrm{ef}$, $D = (1-(h-1)C\tau)^{1/(h-1)}$, $C = 0.75\sqrt{\alpha}\, C_\mathrm{d}$. To natural $\eta > 3$ corresponds series of discreet values of h: $1 < h = \eta/(\eta - 3) \leq 4$. For integer $\eta < 0$ we obtain series of h values, which belongs to interval $0.25 \leq h < 1$ of incomplete regimes of shattering; in this case f_n expresses in $\mathrm{Si}(\varphi)$, $\mathrm{Ci}(\varphi)$. Named set of η values covers compactly enough all the practically important diapason of h. The distributions were calculated for $h = 1.5; 2.0; 4.0$ according to relations (11)–(12). Comparison showed weak influence on $\Delta n(\tilde{r})$ of the two methods – empirical and theoretical – of determination of drop motion law.

8 Conclusions

Model of shattering which is based on mechanism of action of gradient instability provides approximated analytical relations of process: the ablation law and the distribution function of stripped droplets by sizes are obtained. It makes possible to describe quantitatively dynamics of further acceleration and evaporation of spray of daughter droplets and formation of inflammable mixture in wake of shattering drop, and may serve therefore as a ground for model of heterogeneous combustion in detonation wave.

References

[1] Aslanov SK, Girin AG. (1981). About main factors of hydrodynamic instability at the modelling of dispersing process in two-phase detonation. Dokl. of Ukranian Akad. Sci. Ser. A No. 12: 25.

[2] Girin AG. (1985). Hydrodynamic instability and modes of drop shattering. Inzhenerno-fizicheskiy zhurnal (USSR) 48: 771.

[3] Girin AG. (2011). Equations of kinetics of drop shattering in speedy gas flow. Inzhenerno-fizicheskiy zhurnal 84:248

[4] Reinecke WG, Waldman G. D. (1975). Shock layer shattering of cloud drops in reentry flight. AIAA Paper 152: 22 P.