Detonation analogues

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1 Introduction

Analogies have played a prominent role in science throughout its history. Their great value is in cross fertilization of ideas between disciplines which appear at first sight completely unrelated. The purpose of this paper is to point out a range of physical phenomena which bear close relationship with the classical model of detonation developed by Zel'dovich [1], von Neumann [2], and Doering [3] (ZND model), which is a foundation of our current understanding of the basic nature of detonation. Despite the fact that detonation structure is significantly more complex than assumed in the ZND theory, it is basically a shock wave followed by an exothermic reaction zone and in a self-sustained detonation the part of the reaction zone that is causally connected to the lead shock is bounded by a sonic surface. The existence of the sonic point is a key feature of the ZND theory as the sonic conditions provide a necessary closure of the problem, allowing for determination of the detonation speed. Despite its clear importance, the nature of the sonic point is not understood well, especially as one goes beyond the simplified steady-state picture. Detonation theory as a branch of classical fluid mechanics is, unfortunately, considered by general fluid mechanics community as an esoteric subject. This has lead to a lack of communication of important ideas of detonation science to other areas of fluid mechanics. The goal of this work is to present an overview of some examples in fluid dynamics that are analogous in their fundamental aspects to the ZND structure of detonation as a shock wave followed by a transonic flow. In particular, hydraulic jumps in shallow-water flow, traffic jams, particle clogs in two-phase flows, and others will be discussed.

In contrast to Fickett's analogue model of detonation [4], which is essentially a mathematical construct that preserves certain features of the reactive Euler equations, our present discussion is focused on models of real physical phenomena that share certain similarities with detonation. Of course, a complete analogy should not be expected, but certain key aspects are common to detonation and its analogues. The underlying mathematical models of the analogue phenomena share similarity with the reactive Euler equations in that they are also quasi-linear hyperbolic systems with nonlinear source terms, but typically consist of only continuity and momentum equations, which considerably simplifies their analysis. A key common feature between the model systems and reactive Euler equations is the existence of a sonic point in addition to the shock front.

2 ZND theory and its analogues

Let us first review the basic formulation of the ZND theory. A detonation wave is assumed to consist of a shock wave followed by a chemical reaction zone, in which the flow acelerates relative to the shock and at some point becomes sonic, i.e. the local Mach number measured in terms of the particle velocity U relative to the shock, $M = U/c$, becomes unity, $M = 1$ (c is the local sound speed). By manipulating the governing equations, one can arrive at the following equation for the particle velocity [5]:

$$\frac{dU}{dx} = \frac{(\gamma - 1)\, Q\rho c^2 \omega}{c^2 - U^2}, \tag{1}$$

where ω is the rate of reaction, assumed to be a one-step reaction here for simplicity, Q is the heat of reaction, ρ is the density, and γ is the ratio of specific heats. Now the Chapman-Jouguet condition, necessary to determine the detonation speed, is a regularity condition on the particle velocity field: the solution $U\,(x)$ must pass smoothly through the sonic point, $c_c = U_c$. A necessary condition that follows from (1) is that the reaction rate must also vanish at the sonic point, $\omega_c = 0$. These two conditions close the set of governing equations and allow one to determine the detonation speed together with the flow in the reaction zone.

This and other similar, possibly more complex (due to presence of curvature, unsteady effects, several reactions, etc.), arguments are now standard in detonation theory. We also note that there is another, and much more powerful approach to understanding the nature of detonation sonic conditions that is based on the analysis of characteristic surfaces of the reactive Euler equations, but in this paper we limit the discussion to a simple one-dimensional situation where the analysis based on (1) is sufficient.

As it turns out, many natural phenomena involving shock-like structures, are described by equations similar to (1):

$$\frac{du}{dx} = \frac{f\,(x, u)}{c^2 - u^2}, \tag{2}$$

where u is the flow velocity and c is the “sound speed”. The function f depends on a specific phenomenon and involves terms responsible for flow acceleration through the sonic point. Perhaps the first phenomenon that comes to mind when one looks at equation (2) is the textbook example of the isentropic compressible flow in the de Laval nozzle. Recall that the steady-state continuity and momentum equations yield the following equation for the flow speed as a function of the axial distance along the nozzle:

$$\frac{du}{dx} = -\frac{uc^2}{A}\frac{A'}{c^2 - u^2}, \tag{3}$$

where $A\,(x)$ is the slowly-varying area of the nozzle cross section and $A' = dA/dx$. A smooth transition through the sonic point, $u = c$, requires that $A' = 0$ at the same location. Hence a sonic point must be located at the extremum of $A\,(x)$. This same argument can be generalized to more complex situations and to many other phenomena where some kind of a transition through a sonic point occurs. To illustrate, we next discuss two such examples, a hydraulic jump and a traffic jam.

The structure of a circular hydraulic jump is governed by the shallow-water equations, that is the continuity and the radial momentum equations obtained by averaging the Navier-Stokes equations in all other directions except for the radial one [6, 7].

$$\frac{dh}{dr} = \frac{u^2 h/r + gh\,(s_b\,(r) - s_f\,(u, h))}{c^2 - u^2}, \tag{4}$$

where r is the radial distance, h the fluid depth, g the constant of gravity, and u the radial velocity. The remaining terms in the numerator are the slope of the bottom topography $s_b\,(r)$ and the friction slope s_f, which depends on u, h, and ν, the fluid viscosity. One can show that the mere existence of a hydraulic jump is a consequence of the analysis of critical points of (4). The radius of the jump is determined by the shock conditions (conservation of mass and momentum across the shock) and the requirement that the postshock subcritical solution pass through the sonic point, in complete analogy with the ZND solution.

Turning to the example of a traffic flow, we use the continuum approximation consisting of the continuity equation, expressing the conservation of the number of vehicles, and the momentum equation

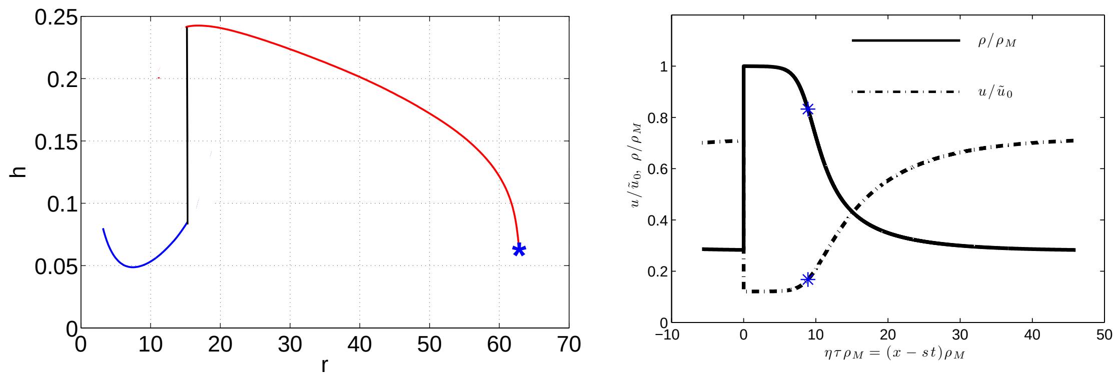

Figure 1: ZND-like structures of a circular hydraulic jump (left, shown is the dependence of the fluid depth, h, on radial distance, r, from the jet impact point) and a traffic jam (right, shown are the normalized traffic density, ρ/ρ_M, and velocity, $u/\tilde{u}_0$.) Stars indicate the sonic point.

which is constructed in direct analogy with momentum equation in fluid mechanics, but reflects key observational aspects of real traffic [7, 8]. With ρ as the number density of vehicles and u as their speed one can show that a travelling-wave solution is described by the differential equation

$$\frac{du}{d\eta} = \frac{(u - s)(u - \tilde{u})}{c^2 - u^2}, \tag{5}$$

where $\eta = x - st$ is a travelling-wave coordinate, s is the speed of the wave (termed "jamiton" in [8],) $\tilde{u}(\rho)$ is the so-called "desired velocity", which is a given function of density, and $c = \sqrt{dp(\rho)/d\rho}$ is obtained from a prescribed law for traffic pressure $p(\rho)$, typically $p = \beta\rho^n$ as in a polytropic fluid. Again, similar to the ZND theory, the jamiton speed is found by requiring that the solution pass smoothly through the sonic point. Typical solution profiles for both the hydraulic jump and the jamiton are shown in Fig. 1. Their striking resemblance of the ZND solution for a detonation wave is clearly seen.

3 Conclusions

It is remarkable that many natural phenomena involving a self-sustained shock wave, whether traveling or standing, are described by a unified theory analogous to the ZND theory of detonation. A self-sustained shock wave always appears to have a post-shock flow which terminates at a sonic point (a critical point of the underlying ordinary differential equation). This feature has, in fact, a simple explanation: a shock wave that is self-sustained must be isolated from the external downstream influence and the sonic point, being a characteristic, serves as an information boundary (event horizon) that provides such isolation. Although, *post factum*, this observation may appear obvious, one must be aware of the fact that these ideas and concepts now familiar to researchers in detonation, are not widely known outside of this field.

There is tremendous value in cross fertilization of ideas between diverse fields of science. In this particular case, experimenting with or analysing the simpler analogue systems can be beneficial for detonation science as detonation studies are difficult and expensive to carry out. Another benefit of the analogue modeling is its potential pedagogical value. Compressible flow does not receive attention it deserves in traditional fluid mechanics courses, not in the least because of its mathematically demanding nature. Nevertheless, a range of phenomena that require such knowledge is extensive and there is great value in advancing the subject of compressible flow and shock dynamics and, in particular, of detonation science.

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