

# Existence and Uniqueness of Travelling Front in Premixed Combustion in Porous Media

Vladimir Gol'dshtein<sup>1</sup>, Natalia Krapivnik<sup>1</sup>,

<sup>1</sup>Ben-Gurion University of the Negev, Department of Mathematics P.O.B 653, Beer-Sheva 84105, Israel

## 1 Introduction

Combustion processes in inert porous media are considered. The inert porous media is filled with combustible gaseous mixture. We focus on a phenomenon of combustion wave driven by a local pressure elevation. This phenomenon represents a relatively new class of problems in the mathematical theory of combustion. We study the travelling wave solutions arising in the adiabatic models that were developed in [1-4]. In the present work we investigate the adiabatic model that includes a quadratic dependence of the friction force on the velocity of the gaseous mixture and which describes the phenomenon of a subsonic pressure-driven flame in inert porous medium. In the frame of reference attached to an advancing combustion wave and after a suitable non-dimensionalization the corresponding mathematical model includes three nonlinear ordinary differential equations (ODEs) with one unknown parameter – flame velocity. An arbitrary solution of the system of ODEs under consideration is represented as a trajectory in the phase space. It is shown that the system has one equilibrium state. The second equilibrium state is introduced artificially (we assume that the chemical reaction level is low for low temperatures with respect to the maximum temperatures of combustion wave). It is demonstrated that the unknown parameter – wave speed – can be selected so that the trajectory that is drawn off from the initial point reaches the stationary point of the system. Moreover, under the reasonable assumption, we show that the pressure driven wave is unique.

## 2 Problem Statement

We describe the main assumptions of the model. We restrict ourselves to a one-dimensional approach. The porous medium is considered as a set of evenly spread parallel capillaries of the same inner radius filled with a premixed combustible gas mixture. The conventional one-temperature approach and cell model is applied. To elucidate the impact of the pressure effect on the wave characteristics the following additional assumption is made. The conventional mechanism of the combustion wave propagation (thermal diffusion) is excluded from our consideration. With the above assumptions the system of governing equations includes six equations: the energy equation (2.1), the concentration equation (2.2), the momentum equation (2.3), the continuity equation (2.4), the equation of the state for ideal gas (2.5), and the Arrhenius reaction rate of chemical reaction (one-step, bimolecular reaction of the first order) (2.6).

$$\frac{d}{dx}(\rho(u - D)c_v T + Pu) = Qw, \quad (2.1)$$

$$\frac{d}{dx}(\rho(u - D)C) = -w, \quad (2.2)$$

$$\frac{d}{dx}(\rho(u - D) + P) = f, \quad (2.3)$$

$$\frac{d}{dx}(\rho(u - D)) = 0, \quad (2.4)$$

$$P = (c_p - c_v)\rho T, \quad (2.5)$$

$$w = AC\rho \exp\left(-\frac{E}{RT}\right). \quad (2.6)$$

The presence of porous media is accounted for by friction force (term  $f$ ) added to the momentum equation (2.3). The expression for friction force is taken in the form which corresponds to the Forcheimer's law

$$f = -K_F \rho u |u|. \quad (2.7)$$

The following notation was used:  $T$  – temperature (K);  $P$  – pressure (Pa);  $C$  – concentration of deficient reactant (kmol/m<sup>3</sup>);  $c$  – specific heat capacity (J/kg/K);  $\rho$  – density (kg/m<sup>3</sup>);  $u$  – gas velocity in the laboratory frame of reference (m/s);  $Q$  – combustion energy (J/kg);  $w$  – reaction rate (kg/s/m<sup>3</sup>);  $K_F$  – permeability of the medium;  $A$  – pre-exponential factor (1/s);  $R$  – universal gas constant (J/kmol/K);  $E$  – activation energy (J/kmol). Subscripts mean:  $p$  – under constant pressure,  $v$  – under constant volume,  $0$  – undisturbed state;  $b$  – burnt (behind the combustion wave front);  $F$  – related to the case of quadratic dependence of the friction force on gas velocity. The system (2.1)-(2.7) is subject to boundary conditions (fresh mixture far ahead of the flame front)

$$T(x \rightarrow +\infty) = T_0 \quad C(x \rightarrow +\infty) = C_0 \quad (2.8)$$

$$P(x \rightarrow +\infty) = P_0 \quad \rho(x \rightarrow +\infty) = \rho_0.$$

The system (2.1)-(2.7) together with the boundary conditions (2.8) is mathematical description of the problem.

To simplify the analysis we introduce the following dimensionless variables

$$\zeta = -\frac{x}{D} \cdot A \cdot \exp\left(-\frac{1}{2\beta}\right), \quad \beta = \frac{R \cdot T_0}{E},$$

$$\eta = \frac{C}{C_0}, \quad \theta = \frac{1}{\beta} \cdot \frac{T - T_0}{T_0}, \quad \Pi = \frac{1}{\beta} \cdot \frac{P - P_0}{P_0},$$

where  $\theta$ ,  $\Pi$ ,  $\eta$  are dimensionless temperature, pressure and concentration, respectively,  $\zeta$  is a dimensionless coordinate, and  $\beta$  is a reduced initial temperature.

### 3 Results

Non-dimensionalization and suitable integration allow us to rewrite the original system (2.1)-(2.8) in the form of the three ODEs

$$\frac{d\theta}{d\xi} = \Lambda_F \frac{(\Pi - \theta)|\Pi - \theta|}{(1 + \beta\Pi)(1 + \beta\theta)} + \varepsilon_1 \eta \frac{1 + \beta\Pi}{1 + \beta\theta} \exp\left(\frac{\theta}{1 + \beta\theta}\right) \quad (3.1)$$

$$\sigma \frac{d\Pi}{d\xi} = \Lambda_F \frac{(\Pi - \theta)|\Pi - \theta|}{(1 + \beta\Pi)(1 + \beta\theta)} \quad (3.2)$$

$$\frac{d\eta}{d\xi} = -\varepsilon_2 \eta \frac{1 + \beta\Pi}{1 + \beta\theta} \exp\left(\frac{\theta}{1 + \beta\theta}\right) \quad (3.3)$$

where the following dimensionless parameters  $\varepsilon_1, \varepsilon_2, \sigma$  are given by

$$\varepsilon_1 = \frac{C_0 \cdot Q}{c_p \cdot T_0 \cdot \beta} \cdot \exp\left(-\frac{1}{2\beta}\right), \quad \varepsilon_2 = \exp\left(-\frac{1}{2\beta}\right), \quad \varepsilon_1 \ll 1, \quad \varepsilon_2 \ll 1,$$

$$\sigma = 1 - \frac{1}{\gamma}, \quad \gamma = \frac{c_p}{c_v}.$$

The expression for the flame velocity reads

$$\Lambda_F = K_F \frac{\beta D^3}{c_p T_0 A} \exp\left(\frac{1}{2\beta}\right).$$

The system (3.1)-(3.2) has an energy integral

$$\eta - 1 + \frac{\varepsilon_2}{\varepsilon_1} (\theta - \sigma\Pi) = 0$$

which allows us to exclude variable  $\eta$ . Thus the system (3.1)-(3.3) reduces to the system of two ODEs. We have shown that the system has one singular point. It is demonstrated that this point is a saddle-knot point. The equilibrium has a one knot sector and two saddle sectors. The knot sector is stable.

In order to prove the existence of the travelling wave solution we used the method that is based on the analysis of phase portrait. Hence, existence of the travelling wave solution is reduced to proving the existence of the path that connects two equilibrium states of the system. In combustion theory, we assume that the chemical reaction level is low for low temperatures with respect to the maximum temperatures of combustion wave. Hence, the first equilibrium state is a stationary state while the second state is introduced artificially. Therefore, we are interested to find  $\Lambda_F$  parameter for which the trajectory  $\Pi(\xi), \theta(\xi)$  leaves

the point  $\Pi(0) = \frac{\varepsilon}{\sigma}, \theta(0) = \varepsilon$  (boundary conditions at the initial stage (fresh mixture)) and

reaches the singular point  $\Pi_b = \frac{\varepsilon_1}{\varepsilon_2(1 - \sigma)}, \theta_b = \frac{\varepsilon_1}{\varepsilon_2(1 - \sigma)}$ . We have proven the existence of

the pressure-driven wave by checking two following conditions for parametric family of paths  $\theta(\Pi, \Lambda_F)$  that are solutions of the equation

$$\frac{d\theta}{d\Pi} = \sigma + \varepsilon_1 \sigma \left(1 - \frac{\varepsilon_1}{\varepsilon_2} (\theta - \sigma\Pi)\right) \frac{(1 + \beta\Pi)^2}{\Lambda_F (\Pi - \theta) |\Pi - \theta|} \exp\left(\frac{\theta}{1 + \beta\theta}\right) \quad (3.4)$$

The first condition is the existence of two  $\Lambda_{F1}, \Lambda_{F2}$  values of the  $\Lambda_F$  parameter (wave velocity) for which the corresponding paths  $\theta(\Pi, \Lambda_{F1}), \theta(\Pi, \Lambda_{F2})$  are located on different sides of the singular point (stationary state)  $(\theta_b, \Pi_b)$ . The second condition is the continuity of the family of paths according to  $\Lambda_F$  parameter.

In order to prove the uniqueness of the fast travelling wave in addition to continuity, we have to show that the equation solution (3.4)  $\theta(\Pi, \Lambda_F)$  is monotonous according to  $\Lambda_F$  parameter. The uniqueness of the wave was proven under additional assumption of the monotonicity of the travelling wave.

## References

- [1] V. Gol'dshtein, I. Shreiber and G. Sivashinsky, On creeping detonation in filtration combustion. *Shock Waves* 4 (1994) 109-112.
- [2] I. Brailovsky, V. Gol'dshtein, I. Shreiber and G. Sivashinsky, On combustion wave driven by diffusion of pressure. *Combust. Sci. Technol.* 124 (1997) 145-165.
- [3] I. Brailovsky and G. Sivashinsky, Hydraulic resistance as a mechanism for deflagration-to-detonation transition. *Combust. Flame* 122 (2000) 492-499.
- [4] I. Goldfarb, V. Gol'dshtein and G. Kuzmenko, Pressure driven flame in porous media. *Phys. Lett. A* 251 (1999) 394-403.