

On the nonlinear dynamics of slowly evolving weakly curved detonation waves

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1 Introduction

Theories of detonation dynamics based on the assumptions of small curvature of the lead shock and slow time evolution, both based on the length and time scale of the steady reaction zone, have been developed over the years by many researchers (see references in [1]). In a recent work [1], Kasimov and Stewart derived an evolution equation for detonation waves with an embedded sonic locus within the approximations of small curvature and slow evolution by reducing the reactive Euler equations using the iterative approximation of Yao and Stewart [3]. One considers time-derivative terms and flow divergence terms in the equations to be small, and ignores the transverse derivatives. The resultant equation (which follows from the forward characteristic equation at the sonic locus) has the following simple form

$$\dot{D} = a_1 \omega_* - a_2 \kappa, \quad (1)$$

where

$$a_1 = \frac{\gamma + 1}{\gamma} \frac{(\gamma^2 - 1) Q D^3}{(1 + D^2)(\gamma + 3D^2)}, \quad a_2 = \frac{\gamma}{\gamma + 1} \frac{(1 + D^2)(D^2 - \gamma)}{\gamma + 3D^2}, \quad (2)$$

ω_* is the reaction rate evaluated at the sonic locus, κ is the shock curvature, D is the normal detonation speed, \dot{D} is the shock acceleration, Q is the heat release, and γ is the adiabatic exponent. The variables are scaled with respect to the initial pressure, p_a , density, ρ_a , $\sqrt{p_a/\rho_a}$, and half-reaction time and length. Since ω_* depends on the depletion factor at the sonic locus, λ_* , one more equation is needed for closure. That equation is the condition of local sonicity, $M_* = 1$, and it can be shown to be [1]:

$$1 + F - \lambda_* + \kappa f(D) + \dot{D} g(D) = 0, \quad (3)$$

where

$$F = (D^2 - D_{CJ}^2) \frac{D^2 D_{CJ}^2 - \gamma^2}{D^2 (D_{CJ}^2 - \gamma)^2}, \quad (4)$$

and $D_{CJ} = \sqrt{\gamma + (\gamma^2 - 1) Q/2} + \sqrt{(\gamma^2 - 1) Q/2}$. The functions $f(D)$ and $g(D)$ are complicated expressions involving several integrals over the quasi-steady planar solution [1]. For a quasi-steady curved detonation, $\dot{D} = 0$, and (1) is a relationship between D and κ . Given sufficiently strong state sensitivity of the heat release rate, $\omega(p, v, \lambda)$, the curve $D(\kappa)$ has an λ shape that connects the points $(D_{CJ}, 0)$ and (c_a, ∞) with two turning points in between (c_a is the ambient sound speed).

Here we extend the analysis of [1] by simplifying the evolution equation and prove that the equation is hyperbolic under all conditions by showing that both functions f and g are always positive. While it is easy to show that f is positive, the fact that g is also positive is harder to prove and was left unproved in [1]. The simplified evolution equation involves only two integrals that can be readily evaluated. The evolution equation can be used to analyze unsteady dynamics of both one-dimensional radially expanding detonations and two-dimensional detonations.

2 Simplified evolution equation

For simple-depletion reaction rate, $\omega = (1 - \lambda)R(p, v)$, by eliminating λ_* from (1) and (3), one obtains the equation

$$\dot{D} + b_1 \kappa + b_2 F = 0, \quad (5)$$

where

$$b_1 = \frac{a_2 + a_0 f}{1 + a_0 g}, \quad b_2 = \frac{a_0}{1 + a_0 g},$$

and f and g can be simplified to the following forms

$$f = \frac{2D}{b^2(1+D^2)} \int_0^{1+F} \frac{p_0(1-v_0)}{\omega_0} d\lambda_0,$$

$$g = g_1 \int_0^{1+F} d\lambda_0 \frac{(1-v_0)}{\omega_0} \left[\frac{1+\gamma^2/D^2}{(\gamma+1)^2} + 2\alpha(1-v_0) + \frac{v_0(v_0-\alpha)}{v_{0*}-v_0} \left(\frac{v_{0*}^2}{v_0^2} - \alpha \right) \right],$$

where, $a_0 = a_1 R(p_{0*}, v_{0*})$, $b = \frac{D(D_{CJ}^2 - \gamma)}{\gamma D_{CJ}(1+D^2)}$, $g_1 = 2D \left[\frac{(\gamma+1)D}{\gamma b(1+D^2)} \right]^2$, $\alpha = (\gamma-1)/(\gamma+1)$, $v_{0*} = \frac{\gamma}{\gamma+1} \frac{1+D^2}{D^2}$, $p_{0*} = \frac{1+D^2}{\gamma+1}$. The integrals here are evaluated using the quasi-steady one-dimensional solution given by

$$p_0 = p_{0*} \left(1 + \gamma b \sqrt{1+F-\lambda_0} \right), \quad v_0 = v_{0*} \left(1 - b \sqrt{1+F-\lambda_0} \right).$$

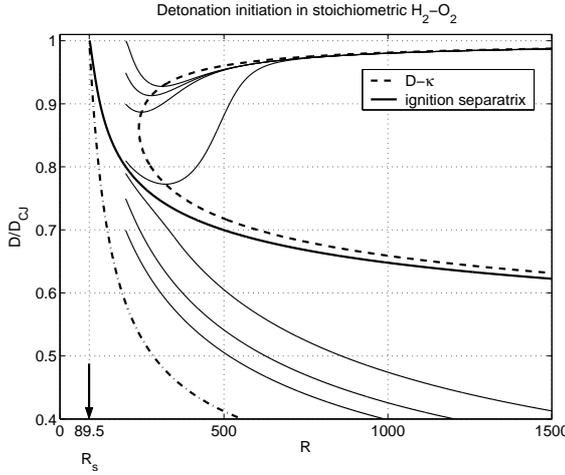


Figure 1: Typical unsteady radially symmetric solutions of (5) starting from various initial conditions (D, R) shown by solid lines [2].

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}_y + \mathbf{b}, \quad (6)$$

where $\mathbf{u} = \begin{bmatrix} D \\ \psi \end{bmatrix}$, $\mathbf{A} = \frac{1}{\sqrt{1+\psi^2}} \begin{bmatrix} D\psi & \frac{b_1}{1+\psi^2} \\ 1 & -D\psi \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_2(D_{CJ} - D) \\ 0 \end{bmatrix}$. The eigenvalues of \mathbf{A} are $\lambda_{1,2} = \frac{D\psi^3}{2(1+\psi^2)^{3/2}} \pm \sqrt{\frac{D^2\psi^6}{4(1+\psi^2)^3} + \frac{b_1+D^2\psi^2}{(1+\psi^2)^2}}$, and positivity of b_1 guarantees that both eigenvalues are real and of opposite sign, thus proving the hyperbolicity of the evolution equation (5).

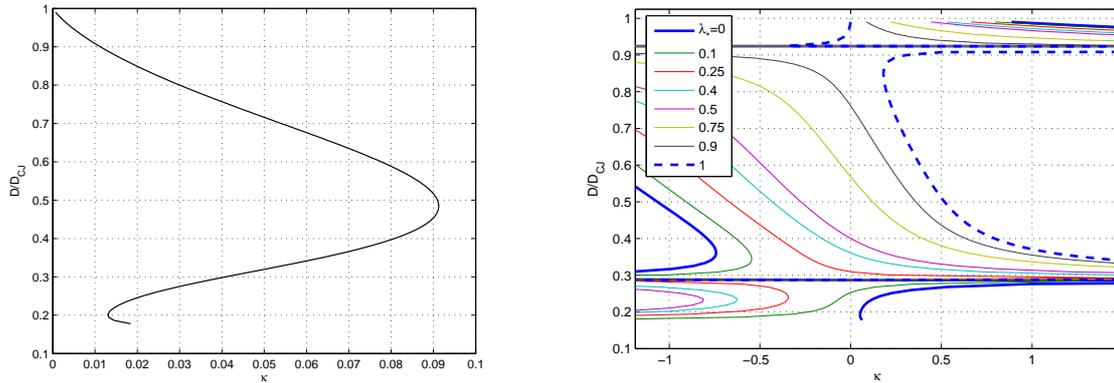


Figure 2: Quasi-steady $D - \kappa$ curve (left) and $D - \kappa$ curves parameterized by λ_* (right) at $E = 10$, $Q = 50$, and $\gamma = 1.2$. Sonic locus exists between the blue solid and dashed lines.

3 Quasi-steady and unsteady dynamics

The evolution equation (5) describes the effects that the energy release and flow divergence have on the shock acceleration. It is a local surface evolution law and is applicable to both one-dimensional and multi-dimensional detonations whose dynamics has the features of slow evolution and weak curvature. In Fig. 2 (left) a typical quasi-steady $D - \kappa$ curve obtained from (5) at $\dot{D} = 0$ is shown for Arrhenius rate law with low activation energy of $E = 10$. At large activation energies typical of gaseous explosive mixtures, the lower turning point is located at very small values of κ . Decreasing E has the effect that the lower turning point moves to larger values of κ and at sufficiently small E the turning points disappear altogether and D becomes a monotonically decreasing function of κ .

For a radially expanding one-dimensional detonation, (5) becomes a second-order ordinary differential equation in the shock radius, $R = n/\kappa$, where $n = 1$ for cylindrical and $n = 2$ for spherical detonations. Typical solutions of this differential equation starting from different initial conditions are shown in Fig. 1, which corresponds to a spherical detonation in a mixture with $\gamma = 1.258$, $E = 38.25$, $Q = 40.5$. As shown in [1, 2], the apparent criticality of solutions as seen in Fig. 1 can be used to estimate critical conditions of direct initiation of gaseous detonation. The dashed line in Fig. 1 is a locus of the quasi-steady $D - \kappa$ curve plotted in $D - R$ plane and the solid lines are the solutions of the evolution equation that start from various initial conditions. The latter are given by the initial radius (fixed) and the initial speed (varied) of the detonation shock. It is found that if the initial speed is above certain critical value (lying on the *ignition separatrix*, thick solid line in Fig. 1), then the solution curves tend to the upper branch of the quasi-steady $D - \kappa$ curve (ignition), while if the initial condition happens to be below the critical curve, the solution curves remain below the ignition separatrix over very long distances (failure).

Equations (1) and (3) do not always have solutions, hence a sonic locus does not always exist (within the limits of the present theory), because the depletion factor λ_* is constrained to be between 0 and 1. By eliminating \dot{D} from (1) and (3), and setting $\lambda_* = 0$ and $\lambda_* = 1$, we obtain two equations relating D and κ that determine the boundaries of the domain of existence of a sonic locus in the theory of self-sustained detonation of [1]. Keeping λ_* as a parameter, we find

$$\kappa(D, \lambda_*) = \frac{F + (1 - \lambda_*)(1 + a_0 g)}{a_2 g - f}.$$

In Fig. 2 (right) we show the boundaries $\lambda_* = 0$ and $\lambda_* = 1$ as well as the $D - \kappa$ curves for several intermediate values of λ_* . The sonic locus can be seen to exist within a large domain which extends from large negative κ through zero to large positive κ . Of course, it is the neighborhood of $\kappa = 0$ which is of significance in this figure, since the theory is based on the limit of small curvature.

4 Conclusions

We have simplified and extended the analysis of [1] of the dynamics of slowly evolving weakly curved detonations. The evolution equation of [1] is proved to be hyperbolic under all conditions, which is an important feature that guarantees the travelling-wave behavior of its solutions. We apply the radially symmetric version of the equation to the analysis of detonation initiation and failure in gaseous mixtures. Application of the equation to the dynamics of two-dimensional detonations is also described.

References

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