Existence Theorem for the Point Source Blast Wave Equation

Shuji Takahashi Department of Mathematical Sciences,

Tokyo Denki Univ., Hatoyama, Saitama,350-0394

We consider in the present paper a point source blast wave to describe the motion of ideal gas initiated by an instantaneous release of finite amount of energy at a point (or line or plane) in a uniform state. Notice that a precise discription of this motion is needed for the hydrodynamic code computation. The explosion starts to expand outwards with its front, headed by a shock wave. The shock wave continuously absorbs ambient air into the blast wave. The equations of motion, continuity and energy of the gas behind a blast wave are written in the form of

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \frac{D\rho}{Dt} = -\rho\left(\frac{\partial u}{\partial r} + \frac{\alpha u}{r}\right), \frac{D}{Dt}p\rho^{-\gamma} = 0. \tag{1}$$

Here u is the particle velocity, p is the presssure, ρ is the density, and $u,\ p,\ \rho$ are functions of the Eulerian coodinate r (measured from the center) and the time t (measured from the instant of explosion). Equations (1) are the Euler equations, which represent a plane wave for $\alpha = 0$, a cylindrical wave for $\alpha = 1$ and a spherical wave for $\alpha = 2$. The derivative D/Dt is defined by $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}$.

The position of the shock front is represented by $R(t)$ (measured also from the center), which is supposed to be a monotonically increasing function of t and is related to the shock velocity U by $\frac{dR}{dt} = U$. On the shock front the conservation laws hold and the discontinuity conditions are given by the Rankine-Hugoniot relations.

The total amount of energy carried by the blast wave is constant and always be equal to the energy released. This condition can be replaced by a stronger condition $u(0,t) = 0$.

Sakurai [1] introduced new independent variables (x,y) defined by $\dfrac{r}{R} = x, \quad \dfrac{C^2}{U^2} = y$, and transformed the dependent variables as $u = Uf(x,y), \quad p = p_0\dfrac{g(x,y)}{y}, \quad \rho = \rho_0 h(x,y)$, The fundamental equations are now transformed into

$$\frac{\lambda}{2}f + (f-x)f_x + \lambda y f_y = -\frac{1}{\gamma h}g_x,\ -\lambda g + (f-x)g_x + \lambda y g_y = -\gamma g\left(f_x + \frac{\alpha f}{x}\right), \tag{2}$$

$$(f-x)h_x + \lambda y h_y = -h\left(f_x + \frac{\alpha f}{x}\right) \tag{3}$$

for x, y in $(0,1)$. The condition on the shock front corresponds to

$$f(1,y) = \frac{2(1-y)}{\gamma+1},\quad g(1,y) = \frac{2\gamma}{\gamma+1}\left(1 - \frac{\gamma-1}{2\gamma}y\right),\quad h(1,y) = \frac{\gamma+1}{\gamma-1}\left(1 + \frac{2}{\gamma-1}y\right)^{-1}. \tag{4}$$

Condition $u(0,t) = 0$ becomes $f(0,y) = 0$. Letting $y = 0$ formally in (2)-(3), the similarity solution is given by $f_0(x) = f(x,0)$, $g_0(x) = g(x,0)$ and $h_0(x) = h(x,0)$, which is now well known. These functions yield the initial conditions.

Our goal is to show the existence of solution for the transformed system (2)-(3) with boundary conditions (4), the initial conditions and condition $f(0,y) = 0$.

The existence is local in y, i.e., in $[0,\bar{y}]$ for some $\bar{y} \in (0,1)$ and the solution is continuous in y, namely, it tends to the similarity solution as $y \to 0$.

References

[1] Sakurai, A (1965) Blast Wave Theory. Basic Development in Fluid Dynamics I, Academic Press: 309–375