Nonlinear Instability of Plane Propagating Flames: Pole Solutions and their Stability

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The nonlinear development that follows after a planar flame front becomes unstable is described, in the context of a weak thermal expansion approximation, by a single nonlinear PDE [1]

$$\varphi_t = \alpha\varphi_{xx} + \frac{1}{2}\varphi_x^2 \; - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\varphi_\xi(\xi,t)}{\xi - x} d\xi \tag{1}$$

where $\varphi(x,t)$ denotes the flame front displacement, x is the spatial coordinate in units of the diffusion length, t is the time in units of the diffusion time, and subscripts denote partial differentiation. The integral term, expressed here as the the Hilbert transform of φ_x, is a linear singular non-local operator $I(\varphi;x)$ which in Fourier space is merely a multiplication by the wavenumber k; i.e. $I(e^{ikx};x) = |k|e^{ikx}$. The physico-chemical parameter α depends primarily on the gas thermal expansion and is responsible for the Darrieus-Landau instability of the planar front. When periodic boundary conditions are assumed, the problem can be expressed in terms of a *single* parameter $\gamma = L/\pi\alpha$ which is proportional to the size of the domain of integration $2L$ and inversely proportional to the gas thermal expansion.

The trivial solution $\varphi = 0$ of equation (1), which corresponds to a plane flame front, is known to be unstable for $\gamma > \gamma_1 \equiv 2$. The evolution of the flame front beyond the stability threshold has been examined by integrating equation (1)numerically [2, 3, 4]. In a typical numerical experiment, one first observes the development of several wrinkles along the flame front that coalesce into one large peak as time progresses. After sufficiently long time, the cusp-like structure appears to propagate at a constant speed without change in shape. The structure of the solution is retained as γ increases except that now more wrinkles appear initially and the depth of the peak, that eventually forms, intensifies and approaches a real cusp as $\gamma \to \infty$.

Some numerical experiments [2, 3], however, suggest that for sufficiently large values of γ the equation displays a qualitatively different behavior: new cusp-like structures appear repetitively on the flame front and the speed of propagation increases indefinitely. It has been tempting to associate this peculiar behavior with the development of secondary structures which have been observed experimentally on the propagating fronts of sufficiently large flames [5, 6]. Whether this behavior is indeed inherent to the PDE, or is a mere product of computational noise, is a question that has been debated in the literature. What has prompted this discussion is the fact that the nonlinear PDE (1) admits exact equilibrium solutions, obtained by a pole decomposition technique and called coalescent pole solutions. The large peak solutions, emerging when the nonlinear PDE is numerically integrated over a long time, appear to belong to the family of these pole solutions. However, the non-steady behavior of the numerical solution of the PDE, presumably observed for large values of γ, does not seem to agree with the expectation from the pole-decomposition theory because the latter does not distinguish between small and large values of γ. Some [4, 7] argue that the PDE is therefore not capable of describing the repetitive generation of new "cusps" when γ is large. The appearance of new "cusps" in the computations results from the limitations of the numerics. To describe mathematically the experimental observation specific to large flames [5, 6] new models need to be derived. Others [8, 9] argue that the inconsistencies with the pole-decomposition theory lie in the stability of the exact pole solutions. The non-steady behavior may be associated with the fact that these equilibrium solutions are unstable in large domains, i.e. when γ is sufficiently large. By numerically solving the initial value problem that results from linearizing the PDE about a pole solution it was concluded that, for large values of γ, pole solutions are unstable. The linear stability of pole solutions was also addressed in [10] using a direct approach. The eigenvalue problem for the perturbed system was formulated and the eigenvalues of the corresponding truncated matrix were determined numerically. In contrast to the results reported earlier, it was concluded here that for any value of γ there exists a stable pole solution.

In this study we address the linear stability of pole solutions and, in contrast to the previous studies, we construct *exact analytical expressions* for the eigenvalues and eigenfunctions. Based on these expressions we make definite statements about the stability question.

The pole decomposition technique formally reduces the PDE (1) to a finite set of ODEs which describe the motion of the poles in the complex plane. There is a natural tendency, implied by these ODEs, for the poles to align parallel to the imaginary axis. The particular set of solutions thus obtained is referred to as the family of *coalescent states* and the equilibrium solutions obtained when the poles are time independent, as the family of *coalescent steady states*. The members of this family are distinguished by the number of poles they possess such that an N-pole coalescent steady state, φ_N with $N = 0, 1, \ldots, \infty$, is a solution made up of N aligned poles. The zero-pole solution corresponds to a flat front. An N-pole solution, of the form

$$\varphi_N(x,t) = U_N t + \hat{\varphi}_N(x) \qquad \text{with } N \geq 1, \tag{2}$$

corresponds to a cusp-like structure that propagates at constant speed,

$$U_N = \frac{N}{\gamma}\left(1 - \frac{2N}{\gamma}\right),$$

without change in its shape. As the value of γ increases the solution becomes more singular and the flame speed increases; a real cusp is developed when $\gamma \to \infty$.

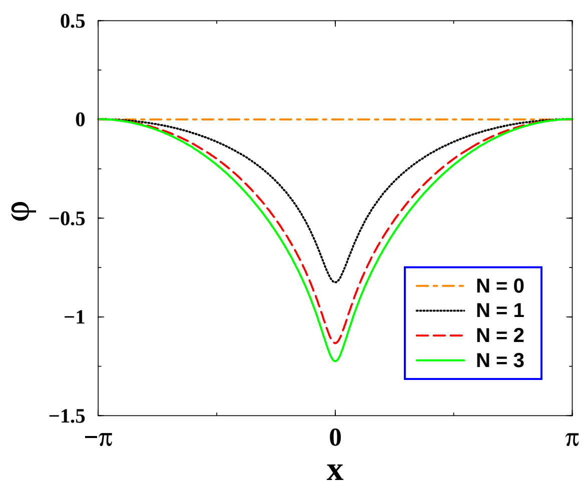

Figure 1: The coalescent steady states for $\gamma = 12$ and different values of N. The direction of propagation here is upwards. Note that the stable solution here is the one corresponding to $N = N_0(12) = 3$.

For a given γ there is an upper bound on the number of poles that a coalescent steady state possesses, namely $N \leq N_0(\gamma)$ with

$$N_0 = \begin{cases} \mathrm{Int}[\gamma/4 + 1/2] & \text{if } \gamma/4 + 1/2 \text{ is not an integer} \\ \mathrm{Int}[\gamma/4 - 1/2] & \text{otherwise} \end{cases}$$

where $\mathrm{Int}[x]$ denotes the greatest integer less than or equal to the real number x. For a given γ and a given $N \leq N_0(\gamma)$ there exists one and only one coalescent steady state. The larger N, the larger the peak of the flame front (see Fig. 1). At the critical point $\gamma = \gamma_1$ at which the flat front becomes unstable, the one-pole solution is born from the zero-pole solution. At the critical point $\gamma = \gamma_2 = 6$, the two-pole solution is born out of the one-pole solution. And, in general (see Fig. 2), the N-pole solution bifurcates from the $(N-1)$-pole solution at $\gamma = \gamma_N = 2(2N-1)$.

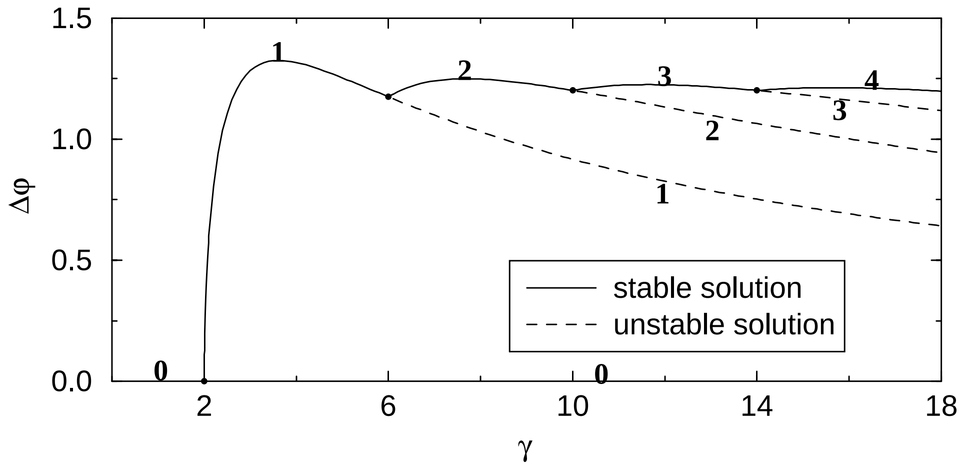

Figure 2: Dependence on γ of the amplitude $\Delta\varphi$ of an N-pole coalescent steady states $\varphi_N(x)$) with $N = 0, 1, 2, 3, 4$, indicating the stable and unstable states. The amplitude is defined as the difference between the maximum and minimum values of φ. The bifurcation points are marked with dots.

We found that the spectrum of the eigenvalue problem associated with φ_N, the N-pole coalescent steady state, consists of two types of eigenvalues:

Eigenvalues of *type I:* There are a finite number of them, all but one are negative. The largest eigenvalue equals zero and corresponds to a lateral translation of the flame front which is of no particular interest.

Eigenvalues of *type II*: There are infinitely many of them; all the eigenvalues are negative when $\gamma - \gamma_N$ is sufficiently small and become positive for larger values of γ. The largest eigenvalue changes sign at $\gamma = \gamma_{N+1} \equiv 2(2N+1)$.

Thus, within the interval $[\gamma_N, \gamma_{N+1}]$ the N-pole solution is asymptotically stable.

The same type of bifurcation that takes place at γ_1, where the flat front (zero-pole solution) becomes unstable, occurs at all other bifurcation points. As γ crosses the bifurcation point γ_N, the largest eigenvalue of the $(N-1)$-pole solution changes its sign from negative to positive. Thus, the $(N-1)$-pole solution becomes unstable while the newborn N-pole solution is stable. For any value of the parameter γ there exists therefore *one and only one asymptotically stable* solution (apart from a trivial translational mode) from the family of coalescent steady states. The asymptotically stable solution is the solution with the largest possible number of poles for this particular value of γ, i.e. $N_0(\gamma)$. Therefore, as the parameter γ increases, the equilibrium states of the PDE undergo a cascade of supercritical bifurcations (see Figure 2).

Being based on analytical expressions, our results resolve unequivocally the earlier controversies that resulted from numerical investigations of the stability problem. Furthermore, the dependence of the eigenvalues and eigenfunctions on the parameter γ provides insight in the behavior of the nonlinear PDE and, consequently, on the nonlinear dynamics of the flame front.

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